Inter-firm Bundling and Vertical Product Differentiation*

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Abstract

This paper studies the competitive effects of bundled discounts offered by pairs of independent firms. In a setting with vertically differentiated goods where firms decide whether to participate in a discounting scheme before prices are set, it is shown that, in equilibrium, all pairs of firms producing goods of the same quality level offer bundled discounts and, relative to the no-bundling benchmark: (i) all headline prices rise; (ii) all bundle prices, net of the respective discount, decrease; and (iii) only high quality sellers will obtain higher profits. Furthermore, this equilibrium corresponds to the worst scenario in terms of consumers’ welfare and decreases social welfare.

Keywords: Bundled Discounts, Bilateral Bundling, Vertical Differentiation.

JEL Classification: D43; L13; L41.

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Introduction

Bundled discounts provide purchasers the opportunity to pay less for a bundle than the sum of the prices of the bundled products when purchased separately. These discounting schemes thus confront consumers with the choice between meeting all their requirements by buying a package at a discounted price and à la carte offerings.

Examples of companies offering bundled discounts include fast food restaurants, telephone companies, book stores, grocery stores and gasoline retailers, to name a few. Despite the fact that bundled discounts are a widespread business practice, the academic literature has devoted limited attention to this issue. So, little is known as to whether this type of discounting schemes should raise anticompetitive concerns.\(^1\)

Bundled discounts affect the market outcome in at least two important ways. On the one hand, they can change the market structure by affecting firms’ incentives to enter or exit the market. On the other hand, they introduce an additional instrument that enables some degree of price discrimination and, consequently, may have an important impact on both consumer surplus and social welfare. Accordingly, one can divide the literature on bundled discounts in two related strands. The first strand has investigated the case in which the bundled discount is offered by a multiproduct firm offering two or more goods or services as a package for a lower price than the aggregate price of its constituent parts. In particular, by examining a setting wherein a monopoly seller in one market faces competition in a second market, a set of recent models (e.g. Peitz (2008), Greenlee et al. (2008) and Nalebuff (2004, 2005)) have shown that the use of bundled discounts can lead to the exclusion of an existing (or potential) equally efficient rival that does not offer an equally diverse group of products. The second strand of the literature analyzes the implications of bundled discounts, per se, on consumer surplus and on social welfare and dates back to Carbajo et al. (1990) and Matutes and Regibeau (1992). Carbajo et al. (1990) show that imperfect competition creates a strategic incentive to bundle which is absent under the polar cases of competitive and monopolized tied markets. Their key argument is that the decision to bundle typically alters the behaviour of rivals in the non-competitive tied market. In their setting, bundling works as a product differentiation device resulting in less aggressive pricing by rivals (competition is relaxed) and lower consumer surplus. Matutes and Regibeau (1992) analyze the behavior of duopolistic firms, both supplying the

\(^1\)As pointed out by Nalebuff (2005, p.364), “[t]he practice of bundle discounts is prevalent, but their effects on competition are not well understood.” Along these lines, Kobayashi (2005) highlights that his “review of the economic literature [on commodity bundling] generally confirms the US Solicitor General’s view in 3M v. LePage’s regarding the underdeveloped state of the economics literature (...).”
two necessary components to make up a system. More specifically, they study the incentives these firms have with respect to: (i) making the components compatible with those produced by the rival firm; and (ii) offering a discount to those consumers who decide to purchase the whole system from them. The main conclusions are that, in most cases, firms will produce compatible goods and offer discounts to those consumers who purchase both components from them. However, there is a prisoner’s dilemma in the sense that firms would have a higher payoff if they could commit not to offer discounts. In addition, when in equilibrium all firms offer discounts, welfare is lower than when there are no discounts.

This pathbreaking work by Matutes and Regibeau (1992) has been extended in several ways. In particular, Thanassoulis (2007) builds on Matutes and Regibeau (1992) by introducing consumers who only value one of the components and by making a distinction between firm-specific and product-specific preferences. As in Matutes and Regibeau (1992), there is a prisoners’ dilemma: firms lose from mixed bundling. Armstrong and Vickers (2010), on the other hand, show that when Matutes and Regibeau’s (1992) model is developed so that consumers need to pay an extra “shopping cost” when purchasing products from more than one firm and consumer preferences for product brands are correlated, then mixed bundling is more likely to lead to welfare gains.

All the above mentioned papers consider the case of bundling by multi-product firms. When this is the case, the discount is given to those consumers who purchase all relevant products from a single firm. However, there are several examples of discounts for bundles where each component good is sold by a different and independent firm. A typical example is the case of supermarket and retail gasoline chains that frequently offer bundled discounts to consumers who purchase from them. These discounts are usually a fixed amount off the headline (or stand-alone) prices that partner firms continue to set independently. Armstrong (2010) presents the following example of bundles in the pharmaceuticals industry: “Pharmaceuticals are sometimes used in isolation and sometimes as part of an approved ‘cocktail’ with one or more drugs supplied by other firms. Drugs companies have the ability to set different prices depending on whether the drug is used on a stand-alone basis or in a cocktail”. (p.2) One such ‘cocktails’ is Atripla, a combination of three HIV medicines: Sustiva, Emtriva and Viread. Sustiva is a registered trademark of Bristol-Myers Squibb while Emtriva and Viread are trademarks of Gilead Sciences, Inc., a different firm. Other examples of bundling between independent firms are the cases of hotels bundling with restaurants, airlines/fast trains with car rental or airlines/fast trains with hotels and public transportation with cultural institutions/leisure, to
name a few. The previous examples refer to cases of discounts that are permanently offered. There are also other examples of discounts that are available for a shorter period of time. Such is the case of gasoline and newspapers/magazines subscription, fast food restaurants and amusement parks, department stores and amusement parks or the joint purchase of different types of alcoholic beverages, such as beer plus Port wine.

The groceries-gasoline example has motivated the work by Gans and King (2006), the most similar previous work to ours. Gans and King (2006) investigate the case where each of the two components in the bundle is produced by two single product independent firms. In their setting, unilateral bundling, i.e., bundling by a single pair of firms, increases the profits of the partner firms to the detriment of the remaining non-bundling firms. Nevertheless, if both pairs of independent firms offer bundled discounts, i.e., if there is bilateral bundling, then each firm’s profits and output end up being the same as in the case where there are no bundled discounts. Moreover, bilateral bundling leads to a social-welfare reduction, as some consumers simply find themselves consuming a sub-optimal branding mix.

Bundling by independent and otherwise unrelated firms differs from bundling by a multi-product firm in the following two ways. First, firms’ objective functions are different in the sense that the multiproduct firm takes into consideration how changing the price of a given product affects the demand for the other products in its portfolio. Second, when the discount is shared by independent firms, there is the need to contract upon its magnitude and also upon how the corresponding cost is divided between the partner firms. This negotiation must occur before the price setting stage and, therefore, there are additional strategic considerations to be dealt with when setting the optimal discount.

One of the assumptions of Gans and King (2006), which is shared by Matutes and Regibeau (1992), Thanassoulis (2007), Peitz (2008) and Armstrong and Vickers (2010), is that products are horizontally differentiated. However, there are several examples of bundled discounts in industries where, at least with respect to one of the products in the bundle, differentiation is clearly vertical. As mentioned, many supermarkets and discount stores offer a grocery-gasoline bundled discount whereby customers receive a discount on their grocery-gasoline purchases

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2Some airline and credit card/hotel/car-rental partnerships are based on accumulating miles and are, therefore, loyalty reward schemes. We are referring here instead to the possibility of getting lower prices when renting a car, conditional on flying with a given airline: in many instances, by presenting the boarding passes and a discount code, travelers will be given a discount.

3Other work that focuses on bundled discounts by independent producers is Maruyama and Minamikawa (2009), who study the incentives for vertical integration. Also, Armstrong (2010) discusses the impact on profits, consumer surplus and welfare induced by inter-firm discounts, focusing on independent, partially substitutable and complementary products.
from the firms involved in the discounting scheme. It should be noted, however, that in this example, there is vertical differentiation, at least, on the groceries’ side: most consumers consider that supermarkets chains and discount stores offer products and services of different quality. Another case in point refers to bundled discounts offered to consumers who look for airline tickets and car-rental. Low cost airlines and national carriers are also, in most cases, vertically differentiated, as well as local rent-a-car versus multinational rent-a-car agencies. Vertical differentiation, at least as far as consumer perception is concerned, is also present in the pharmaceutical industry, with the distinction between branded products and generics. Needless to say, hotels and restaurants are also vertically differentiated. To the best of our knowledge, however, vertical differentiation has been neglected by the extant literature on bundled discounts.

In this paper, we contribute to cover this gap in the literature by proposing a model of strategic interaction between four producers of two different and unrelated products to study the likely competitive effects of bundled discounts in the presence of vertical differentiation. We depart from Gans and King (2006) in two ways. First, as mentioned, we consider a different type of product differentiation. Second, while Gans and King (2006) assume that all consumers are interested in buying both available products, in this paper we assume, following Thanassoulis (2007), that some consumers are only interested in buying one of the products while others desire to buy both available products. In particular, we assume that consumers interested in both products are arrayed on a unit square where each axis measures consumers’ valuations for quality of the two products. Each good is produced by two firms, a high quality producer and a low quality producer. Pairs of firms may then agree to offer jointly a bundled discount across the two products and to share the costs of that discount. We address, apart from the no discounting benchmark case, five different scenarios: (i) unilateral bundled discount by the low quality firms; (ii) unilateral bundled discount by the high quality firms; (iii) bilateral bundling by firms with similar quality levels; (iv) unilateral bundled discount by a high and a low quality firm; and (v) bilateral bundling by firms with different quality levels.

This theoretical framework enables us to raise a number of interesting questions: What are the welfare effects of bundled discounts in each of the five scenarios described above? Are bundled discounts consumer-surplus-enhancing? Should bundled discounts be free of antitrust concerns in this context? Which firms have the highest incentives to offer the discounts? Does the prisoners’ dilemma identified in the literature carry over to the case of vertical differentiation? The answer to these and other related questions is, to our knowledge, not yet known and is the main focus of this paper.
We start by studying the competitive effects of the introduction of bundled discounts in each of the five scenarios identified above. Our main results are the following. First, and relative to the no-discounting benchmark case, the headline prices of the bundling firms always rise. Second, bundled discounts may induce a decrease in consumer surplus and always induce a reduction in total welfare. This leads to the conclusion that in none of the five scenarios should bundled discounts be free of antitrust concerns.

We then turn to the study of the firms’ decisions regarding their eventual participation in a bundled discount scheme. Here, we also depart from Gans and King (2006). In their setting, the identity of the firms that offer the discount is irrelevant due to symmetry assumptions. The same is not true, however, in a setting like ours where there is vertical differentiation, implying that the quality levels of the products involved in the bundle will have an impact on firms’ bundling decisions. Under several modelling choices regarding how coalitions of firms involved in bundled discount schemes are formed, the (robust) outcome of this coalition formation game is that of bilateral bundling by producers offering goods of the same quality level. However, and in contrast with Matutes and Regibeau (1992), Gans and King (2006), Thanassoulis (2007) or Maruyama and Minamikawa (2009), in this bilateral bundling scenario the high quality firms earn higher profits than in the status quo no-discounting situation and, therefore, firms do not find themselves in a prisoner’s dilemma situation: allowing for vertical differentiation results in the elimination of the Bertrand bundling super-trap identified by Gans and King (2006). Nevertheless, this scenario is shown to be the one leading to the most adverse consequences in terms of consumer welfare and also leads to a reduction of social welfare. This then suggests that competition authorities should scrutinize in detail bundling discounting by independent producers of vertically differentiated goods.

The remainder of the paper is organized as follows. In Section 2, we lay down our general framework and specify the timing of the proposed game. In Section 3, we study the competitive effects of bundling in all scenarios considered. Section 4 studies the coalition formation stage, a simultaneous or sequential move game where firms decide upon its participation in a bundled discount scheme. Section 5 investigates the robustness of the main results. Finally, Section 6 concludes the paper. A sketch of the proofs and some longer expressions are relegated to the Appendix.

4The prisoner’s dilemma is a recurrent result in the price discrimination literature. As Armstrong (2008) highlights, “(...) an oligopolistic firm is always better off if it can price discriminate compared to when it cannot, for given prices offered by its rivals. However, as in many instances of strategic interaction, once account is taken of what rivals too will do, firms in equilibrium can be worse off when price discrimination is permitted. Firms then find themselves in a classic prisoner’s dilemma.”
2 The model

2.1 Firms

We consider the case of two distinct products, X and Y, each sold by two firms, a high quality producer and a low quality producer. Denote by \(A_X, A_Y\) the two high quality producers and by \(B_X, B_Y\) the two low quality producers of products X and Y, respectively. There are no costs associated with the production of either product or quality level. We denote the price of the higher quality product by \(P_i\) and the price of the lower quality product by \(p_i\), with \(i = X, Y\). In what follows, \(s_j\) represents an index of the quality of the product sold by firm \(j\), and \(\Pi_{ji}\) the profit of firm \(ji\), with \(j = A, B\) and \(i = X, Y\). Let the quality difference \(s_A - s_B > 0\) be denoted by \(s\).

Each pair of producers of different products may agree to participate in a bundled discount scheme. In all, we consider six different scenarios. Scenario 0 is the benchmark case in which there are no discounts. Scenarios 1 and 2 are unilateral bundling scenarios and refer, respectively, to the cases of a discount given by the low quality or by the high quality firms. Scenario 3 refers to the case of bilateral bundling by firms of the same quality level, i.e. simultaneous bundled discounts offered by both the low and the high quality pairs of firms. Scenarios 4 and 5 consider the cases of unilateral and bilateral bundling by firms of different quality levels, respectively. Let \(\gamma_{ji}\), with \(j = A, B, i = A, B\) denote the discount offered by the producers of \(j_X\) and \(i_Y\). Throughout the paper and without loss of generality, we will write \(\gamma_{ji}\) as \(\beta_{ji}s\). When \(\beta_{ji} > 0\) we say that firms \(j_X\) and \(i_Y\) are partner firms in the discounting scheme. For instance, when the two high quality producers are partners, consumers that purchase the high quality bundle will pay \(P_X + P_Y - s\beta_{AA}\).\(^5\) Moreover, we denote by \(\alpha_{ji}\) the percentage of the discount financed by the producer of X and whenever \(i = j\) we will write \(\beta_{ji}\) and \(\alpha_{ji}\) merely as \(\beta_j\) and \(\alpha_j\).

As Gans and King (2006), we are interested in the profitability of relatively small discounts. This being the case, for each scenario, we define a small discount as a discount that does not exceed 50% of the average price of the two products in the bundle, evaluated at the no bundle equilibrium. Such a discount implies that the equilibrium in the pricing stage is such that the four possible combinations of products have a positive market share for all possible ways of sharing the discounts and for all discount levels of the competitors (if any).\(^6\)

\(^5\)As Armstrong (2010) points out, this additive bundled discount is “probably more easily implemented in practice relative to a system of choosing a rigid bundle price (...) and then negotiating how to share that revenue.”

\(^6\)In section 5 we show that the results are robust to other definitions of small discounts, provided that the
Our assumption of small discounts is justified by some management literature that points to the ineffectiveness of high discounts. Raghubir (1998) illustrates that consumers use the coupon value (either percentage discount or cents-off coupon) to form their expectations about price. A high discount then signals a high price and, therefore, may make purchase less likely. Barat and Paswan (2005), using questionnaire data, conclude that, “for low face values of coupon, intention to redeem is positively associated with face value, whereas, for the higher face values of the coupon, the intention remains more or less unchanged. The correlation between intention to redeem the coupon and the perceived sticker price of the product is positive at the lower levels of coupon face value, but becomes negative for higher face values.” Raghubir (2004) analyzes the contexts that lead consumers to make negative inferences about price or quality of a product from coupon discounts. One of the implications of her results is that “doubling coupon values may neither profitably, nor reliably increase a product’s sales”. Along different lines, the management literature has also addressed how the perception of price unfairness affects consumer behavior and, ultimately, sellers’ profits. The perception of unfairness may arise from price discrimination, as recently analyzed in Wu et al. (2011) whose results “show that post-purchase disclosure of discrimination information elicits higher negative emotions for indirect discrimination involving coupon and purchase quantity, but is rather inconsequential for direct discrimination or indirect discrimination through membership.”

It should also be highlighted that small discounts are empirically more relevant. In fact, gasoline discounts usually amount to 5 to 10 Euro cents whereas restaurant discounts for guests at hotels are usually at around 10%. As for the HIV medicines mentioned above, online information states that “Atripla is cheaper ($1,465 vs $1,479 per 30 days’ supply) than if the three components are purchased individually”, which represents a discount of around 1% on wholesale prices.7

This assumption also makes the analysis more tractable as it limits the number of cases that one has to address. Additionally, the constraint on the discount level that results from this assumption is only binding in one of the five scenarios considered.

2.2 Consumers

The way we model consumers’ preferences for quality follows Gabszewicz and Thisse (1979). Consumers purchase at most one unit of each good and we assume, following Thanassoulis (2007), that some consumers are only interested in buying one of the products (X or Y) while four combinations of products have a positive demand.

7http://www.ncbi.nlm.nih.gov/pmc/articles/PMC2504066/
others desire to buy both products. In addition, for simplicity, throughout the paper we assume that there exist three equal sized groups of consumers: the number of consumers who value only good $X$ is equal to the number of consumers who value only good $Y$ and also equal to the number of consumers who value both goods.$^8$

Consumer $k$’s net utility when purchasing product $i$ from producer $A_i$ is given by $V_i + \theta_i^k s_{A_i} - p_i$, whereas consumer $k$’s net utility when purchasing product $i$ from producer $B_i$ is given by $V_i + \theta_i^k s_{B_i} - p_i$, with $i = X, Y$. In one of the three groups of consumers, $V_i$, $i = X, Y$, is assumed to be sufficiently high so that every consumer purchases one unit of each good. In the other two consumer groups, however, either $V_i$ or $V_j$ is sufficiently small, meaning that consumers in these groups only desire to buy one of the available products. We also assume that the valuations of consumers regarding the products they value is sufficiently high so that the market is fully served. That is, all consumers interested in both products will buy one unit of each product, and consumers who are in the market for one product only will buy one unit of that product (only).

Consumers are assumed not to get any extra benefit or any transaction costs reduction from purchasing a specific pair of products. Hence, in the absence of a discount, the demand for one product is independent from the demand for the other. Further, we also assume that for those consumers who value both products, valuations for the quality, $(\theta_X^k, \theta_Y^k)$, are uniformly distributed in $[0, 1] \times [0, 1]$. In addition, for those consumers who value only one of the available products, valuations for this product are uniformly distributed along a line of unit length.

Finally, when we say that a given consumer purchases, say, $A_X, B_Y$ we mean that this consumer purchases the high quality version of good $X$ and the low quality version of good $Y$. We denote the quantity of such pair of products by $Q_{A_X, B_Y}$. As for the consumers who care for only one product, the quantity demanded of, say, the high quality version of product $X$ is denoted by $K_{A_X}$.

### 2.3 Timing

The timing of the game played between firms is divided in two stages: (1) a coalition formation stage and (2) a competition stage.

In the first stage, firms play a coalition formation game, described in detail in section 4, from where the identities of the partner firms that finance the discount is obtained. At this stage, we assume that each firm simultaneously selects another firm (if any) that it wants to

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$^8$In contrast, both Matutes and Regibeau (1992) and Gans and King (2006) assume that all consumers in the market must purchase both available goods in order to derive any positive utility.
offer a bundled discount with. In case of coincidence of matching intentions, a “coalition” is formed, i.e., the pair of firms with matching intentions will offer a bundled discount together. Afterwards, prices and discounts are set at the competition stage, which can be further divided into two stages: First, firms agree to their bundled discount, if any, and to the way of sharing it. Second, firms set their headline prices simultaneously.

So, partner firms choose both the discount and the percentage of the discount financed by each producer so as to maximize their joint profits.\(^9\) Afterwards, each single-product firm sets (noncooperatively and simultaneously) its headline price. Since the discount is set before the headline prices, we are implicitly assuming that it is easier for a firm to change its own price than to change the bundled discount. This is a natural assumption given that any firm is free to unilaterally change its headline price at a short notice, whereas changing the discount would involve a renegotiation with the partner firm. Moreover, it is often the case that firms advertise the discounting scheme but not the headline prices. This is a way of committing in advance to the agreed upon discount while still allowing for future changes in the headline prices.

3 Competition Stage

In this section, we investigate the consequences of bundled discounts relative to the benchmark situation where there are no discounts. More specifically, we analyze and discuss the six possible outcomes of the coalition formation stage. In what follows, we start by presenting the relevant demand functions and then investigate each of the six different scenarios.

3.1 Preliminary Results

The following lemma presents the relevant demand functions. Let 

\[ \theta_i^{AA} := \frac{P_i - p_i - s_{\beta A}}{s}, \quad \theta_i^{BB} := P_i - p_i + s_{\beta B}, \quad \theta_V^{YB} := P_Y - p_Y - s_{\beta AB}, \quad \theta_X^{XB} := P_X - p_X - s_{\beta AB}, \quad \theta_Y^{YA} := P_Y - p_Y - s_{\beta BA}, \quad \theta_X^{AX} := P_X - p_X + s_{\beta BA}, \]

with \( i = X, Y \).

Lemma 1:

(i) Consider the case where there are discounts given by firms with the same quality. Assume

\(^9\)This can be thought of a joint organization setting the discount on behalf of the two firms involved.
that \((\theta_Y^{AA}, \theta_X^{AA}, \theta_Y^{BB}, \theta_X^{BB}) \in [0,1]^4\). Then:

\[
Q_{AX, BY} = \left(1 - \frac{P_X - p_X + s\beta_B}{s}\right) \left(\frac{P_Y - p_Y - s\beta_A}{s}\right),
\]
\[
Q_{BX, AY} = \left(1 - \frac{P_Y - p_Y + s\beta_B}{s}\right) \left(\frac{P_X - p_X - s\beta_A}{s}\right),
\]
\[
Q_{AX, AY} = \left(1 - \frac{P_Y - p_Y - s\beta_A}{s}\right) \left(1 - \frac{P_X - p_X - s\beta_A}{s}\right) - \frac{(\beta_A + \beta_B)^2}{2},
\]
\[
Q_{BX, BY} = \left(\frac{P_Y - p_Y + s\beta_B}{s}\right) \left(\frac{P_X - p_X + s\beta_B}{s}\right) - \frac{(\beta_A + \beta_B)^2}{2}.
\]

(ii) Consider the case where there are discounts by firms with different quality. Assume that
\((\theta_Y^{AB}, \theta_X^{AB}, \theta_Y^{BA}, \theta_X^{BA}) \in [0,1]^4\). Then:

\[
Q_{AX, BY} = \left(1 - \frac{P_X - p_X - s\beta_{AB}}{s}\right) \left(\frac{P_Y - p_Y + s\beta_{AB}}{s}\right) - \frac{(\beta_{AB} + \beta_{BA})^2}{2},
\]
\[
Q_{BX, AY} = \left(1 - \frac{P_Y - p_Y - s\beta_{BA}}{s}\right) \left(\frac{P_X - p_X + s\beta_{BA}}{s}\right) - \frac{(\beta_{AB} + \beta_{BA})^2}{2},
\]
\[
Q_{AX, AY} = \left(1 - \frac{P_Y - p_Y + s\beta_{AB}}{s}\right) \left(1 - \frac{P_X - p_X + s\beta_{BA}}{s}\right),
\]
\[
Q_{BX, BY} = \left(\frac{P_Y - p_Y - s\beta_{BA}}{s}\right) \left(\frac{P_X - p_X - s\beta_{AB}}{s}\right).
\]

(iii) Consider the group of consumers who only value one product. The corresponding demand functions for each quality level of product \(i, i = X, Y\), are given by:

\[
K_{A_i} = \left(1 - \frac{P_i - p_i}{s}\right)
\]
\[
K_{B_i} = \left(\frac{P_i - p_i}{s}\right)
\]

The different thresholds \(\theta_Y^{AA}, \theta_X^{AA}\), etc. represent the consumers that are indifferent between two of the four alternatives available. For instance, \(\theta_{ij}^k\) represents those consumers indifferent between the pair \(i_X, j_Y\), which entitles them to a discount \(\beta_{ij}\), and the pair that has different quality in product \(k\).

Making use of Lemma 1 and of the assumption that the number of consumers interested in buying product \(X\) only, product \(Y\) only or both products is the same, the demand function for each individual product can be obtained from:

\[
Q_{AX} = Q_{AX, BY} + Q_{AX, AY} + K_{AX} \quad \text{and} \quad Q_{BX} = Q_{BX, AY} + Q_{BX, BY} + K_{BX}
\]
\[
Q_{AY} = Q_{BX, AY} + Q_{AX, AY} + K_{AY} \quad \text{and} \quad Q_{BY} = Q_{AX, BY} + Q_{BX, BY} + K_{BY}.
\]

Clearly, all quantities are a function of prices and discounts. For the sake of brevity, however, we write \(Q_{AX, BY}(P_X, p_X, P_Y, p_Y, \gamma_A, \gamma_B)\) as \(Q_{AX, BY}\) and so forth.
3.2 Bundling by firms with the same quality level

Assume that the high quality pair of firms and the low quality pair of firms offer discounts given, respectively, by $\beta_A$ and $\beta_B$ and that the percentage of the discount financed by the producer of $X$ is given by, respectively, $\alpha_A$ and $\alpha_B$. Equilibrium prices result from the individual maximization of the following objective functions:

$$
\Pi_{AX} = P_X (Q_{AX,BY} + Q_{AX,AY} + K_{AX}) - \alpha_A \beta_A s Q_{AX,AY}
$$

$$
\Pi_{AY} = P_Y (Q_{BX,AY} + Q_{AX,AY} + K_{AY}) - (1 - \alpha_A) \beta_A s Q_{AX,AY}
$$

$$
\Pi_{BX} = p_X (Q_{BX,AY} + Q_{BX,BY} + K_{BX}) - \alpha_B \beta_B s Q_{BX,BY}
$$

$$
\Pi_{BY} = p_Y (Q_{AX,BY} + Q_{BX,BY} + K_{BY}) - (1 - \alpha_B) \beta_B s Q_{BX,BY}
$$

The expressions for the equilibrium prices are presented in Appendix B.

3.2.1 Scenario 0: No discounting benchmark case

Consider first the benchmark case where $\beta_{ji} = 0$ for all $i$ and $j$. In the absence of discounts, the decision to purchase a high or low quality version of one product is independent of the prices of the two different quality variants of the other product. Purchasing choices regarding the group of consumers interested in both products in this no-discounting equilibrium are illustrated in Figure 1.

Lemma 2: If no pair of firms offers a bundled discount, then, in equilibrium:

(i) the headline prices for the high quality firms will be $P_X = P_Y = 2s/3$.

(ii) the headline prices for the low quality firms will be $p_X = p_Y = s/3$.

(iv) consumer surplus will be $CS_0 = \frac{2}{9} (11s_B - 2s_A)$.

(v) the individual profit of the high quality firms will be $\Pi_{AX} = \Pi_{AY} = 8s/9$ and the individual profit of the low quality firms will be $\Pi_{BX} = \Pi_{BY} = 2s/9$.

(vi) welfare will be $TW_0 = \frac{2}{9} (8s_A + s_B)$.

3.2.2 Scenario 1: Unilateral bundling by the low quality firms

In this section, we consider the effects of bundled discounts by the pair of low quality producers, assuming that the pair of high quality producers does not offer a discount. Hence, $\beta_B > 0$ while $\beta_A = 0$. The resulting division of consumers interested in buying both products is as in Figure 2.
Figure 1: Consumers’ choices with no discount.

Figure 2: Consumers’ choices with bundled discounts by firms with the same quality level.
Making use of Appendix B, equilibrium prices for the high quality pair of firms are given by

\[
P_X(\alpha_B, \beta_B) = P_Y(1 - \alpha_B, \beta_B) = \frac{\left(\alpha_B^2\beta_B^2 (\beta_B + 2)^2 + \alpha_B\beta_B (16\beta_B - 12\beta_B^2 - 3\beta_B^3 + 8) + 8 (\beta_B^3 - 3\beta_B^2 - \beta_B + 12)\right) s}{4 \left(36 - \alpha_B (1 - \alpha_B) \beta_B^2 - 6\beta_B^2\right)}.
\]

As for the low quality pair of firms, equilibrium prices are:

\[
p_X(\alpha_B, \beta_B) = p_Y(1 - \alpha_B, \beta_B) = \frac{\left(\alpha_B^2\beta_B^2 (8\beta_B + \beta_B^2 + 4) + \alpha_B\beta_B (40\beta_B - 12\beta_B^2 - 3\beta_B^3 + 16) + 8 (2 - \beta_B) (2\beta_B + \beta_B^2 + 3)\right) s}{4 \left(36 - \alpha_B (1 - \alpha_B) \beta_B^2 - 6\beta_B^2\right)}.
\]

The optimal discount level and the discount sharing rule are then obtained by maximizing the low quality firms’ joint profit, \(\Pi_B(\alpha_B, \beta_B) = \Pi_{B_X}(\alpha_B, \beta_B) + \Pi_{B_Y}(\alpha_B, \beta_B)\), evaluated at the equilibrium prices above, with respect to \(\beta_B\) and \(\alpha_B\). For any admissible \(\beta_B\), this results in \(\alpha_B = 1/2\). Now, when deciding upon the optimal discount level, the low quality firms will take into consideration the effects that different discount levels will have on the equilibrium prices.

These effects can be described as follows. All else constant, a higher discount offered by the low quality producers increases the demand for the bundle, which means that demand for each low quality component increases. Also, the introduction of the discount can be interpreted as a unit cost, partially incurred by each partner firm, for the units entitled to the discount. As a result of the higher demand for the low quality goods, on the one hand, and of the higher costs incurred by firms involved in the discounting scheme (the low quality producers), on the other, headline prices for the goods in the bundle increase with the discount. Despite the fact that both (headline) low quality prices increase with the discount, their sum increases at a lower rate than the discount itself and, as a result, the “net” bundle price decreases with the discount. As for the high quality firms, a higher discount given by the low quality producers gives rise to two different effects: a lower demand for their products, on the one hand, but also that their competitors will face “higher costs”, on the other. The first effect dominates the second one and, as a result, the high quality headline price is decreasing in \(\beta_B\). The sum of the headline prices of two different quality products increases with the discount. As the high quality product headline price decreases in \(\beta_B\) at a rate that is more than compensated by the low quality product headline price increase in \(\beta_B\), consumers purchasing a pair of products of different quality will end up paying a price that increases with the discount.

Maximization of \(\Pi_B(\alpha_B^*, \beta_B)\) leads to \(\beta_B^* = 0.11136\), which is obtained numerically and respects our small discount restriction. This optimal discount results from the trade-off between
several effects, which we discuss in turn. The discount impacts the low-quality firms’ aggregate profit directly but also strategically, via the equilibrium prices. The *direct effect* is given by:

\[
\frac{d\Pi_B}{d\beta_B} = \frac{\partial \Pi_B}{\partial Q_{BX,AY}} \frac{\partial Q_{BX,AY}}{\partial \beta_B} + \frac{\partial \Pi_B}{\partial Q_{AX,BY}} \frac{\partial Q_{AX,BY}}{\partial \beta_B} + \frac{\partial \Pi_B}{\partial K_{BX}} \frac{\partial K_{BX}}{\partial \beta_B} + \frac{\partial \Pi_B}{\partial K_{BY}} \frac{\partial K_{BY}}{\partial \beta_B} + \frac{\partial \Pi_B}{\partial \beta_B}.
\]

The direct effects on demand by consumers interested in both products are illustrated in Figure 3, where the grey area represents the increase in the market share of the low quality bundle.

This direct effect works as follows. When the discount increases, some consumers purchasing \(B_X, A_Y\) and \(A_X, B_Y\), as well as \(A_X, A_Y\), will switch to the bundle \(B_X, B_Y\). Hence, the demand for the low quality firms will increase. However, all those consumers that were previously purchasing \(B_X, B_Y\) will now benefit from the increased discount.

Evaluated at the no-discount equilibrium, the net direct effect is positive and boils down to:

\[
\frac{1}{3}s \left( -\frac{1}{3} \right) + \frac{1}{3}s \left( -\frac{1}{3} \right) + \frac{2}{3}s \left( \frac{2}{3} \right) + 0 + 0 - s \left( \frac{1}{9} \right) = \frac{1}{9}s
\]

If the discount level and the prices were set simultaneously, as in most of the previous literature on bundling by a multi-product firm, the *direct effect* would characterize completely the incentives to offer discounts. However, in the case of single-product independent and otherwise unrelated partner firms, there is a pre-commitment to a discount and prices are set at a later stage. This implies that, when setting the discount, firms must also consider the
impact this discount will have both on the price set by the non-partner firms as well as its impact on the price charged by the partner firms.

Let us first discuss the effect on the partner firms’ profits that is due to the change in the high quality prices in response to an increase in the discount. As this results from changes in the prices of the non-partner firms, we call this the \textit{external strategic effect}:

\[
\sum_{i=X,Y} \frac{\partial (\Pi_{B_X} + \Pi_{B_Y})}{\partial P_i} \frac{\partial P_i}{\partial \beta_B} = \\
= \sum_{i=X,Y} \left(p_X \frac{\partial (Q_{B_X,A_Y} + Q_{B_X,B_Y} + K_{B_X})}{\partial P_i} + p_Y \frac{\partial (Q_{A_X,B_Y} + Q_{B_Y,B_Y} + K_{B_Y})}{\partial P_i} - \beta_B \frac{s}{2} \frac{\partial Q_{B_X,B_Y}}{\partial P_i} \right) \frac{\partial P_i}{\partial \beta_B}.
\]

An increase in the discount will induce a decrease in the headline prices for the high quality products. As a result, some consumers will switch from \(B_X,B_Y\), \(B_X,A_Y\) and \(A_X,B_Y\) to \(A_X,A_Y\), while others will switch from \(B_X,B_Y\) to \(B_X,A_Y\) or to \(A_X,B_Y\). This will result in a decrease in the quantity demanded from the low quality firms but also in a reduction in the number of customers entitled to the discount. Despite this trade-off, the effect has a negative impact on the partner firms’ joint profit and, therefore, mitigates the effect on that joint profit induced by the previous (direct) effect. Evaluated at the no-discount equilibrium, this external strategic effect boils down to:

\[
\left(\frac{1}{3} s \left(\frac{2}{s}\right) + \frac{1}{3} s (0)\right) \left(-\frac{2}{36^s}\right) = -\frac{1}{27^s}
\]

Finally, we present the (cross) effect in the partner firms’ profits that is due to the change in the low quality products’ prices in response to an increase in the discount. We call this the \textit{internal strategic effect}:\footnote{This effect would not exist in the case of bundling by a multiproduct firm. In that case, both prices and the discount are set to maximize the same objective, the profit of the low quality firm. Hence, if there was a merger between the partner firms without any impact on the timing of the game, this effect would disappear.}

\[
\frac{\partial \Pi_{B_X}}{\partial p_X} \frac{\partial p_X}{\partial \beta_B} + \frac{\partial \Pi_{B_X}}{\partial p_Y} \frac{\partial p_Y}{\partial \beta_B} = \\
= \left(p_Y \left(\frac{\partial Q_{A_X,B_Y}}{\partial p_X} + \frac{\partial Q_{B_X,B_Y}}{\partial p_X} + \frac{\partial K_{B_Y}}{\partial p_X}\right) - \frac{\beta_B}{2} \frac{s}{2} \frac{\partial Q_{B_X,B_Y}}{\partial p_X}\right) \frac{\partial p_X}{\partial \beta_B} + \\
+ \left(p_X \left(\frac{\partial Q_{B_X,A_Y}}{\partial p_Y} + \frac{\partial Q_{B_X,B_Y}}{\partial p_Y} + \frac{\partial K_{B_Y}}{\partial p_Y}\right) - \frac{\beta_B}{2} \frac{s}{2} \frac{\partial Q_{B_X,B_Y}}{\partial p_Y}\right) \frac{\partial p_Y}{\partial \beta_B}.
\]

An increase in the discount will lead to an increase in the headline prices for the low quality products. The low quality producer of product \(Y\), firm \(B_Y\), will be affected by the increase in the headline price of the low quality version of product \(X\) in the following way. Some consumers previously purchasing \(B_X,B_Y\) will switch to \(A_X,B_Y\), leaving the total demand of
the low quality producer of $Y$ unchanged.\textsuperscript{11} This has a positive effect which is linked to the reduction of customers entitled to the discount. However, at the no-discount equilibrium, this last effect does not change the profit of the low quality producer of $Y$.\textsuperscript{12} Likewise for the low quality producer of $X$.

As noted above, the two latter effects only exist when the discount is set in advance with respect to prices. For small discounts, the sum of these two last effects is negative and, hence, the optimal discount in our setting is lower than the one corresponding to the case in which price and discount decisions are determined simultaneously.\textsuperscript{13}

The following lemma summarizes our results regarding unilateral bundling by the low quality firms:

**Lemma 3:** If only the low quality firms offer the bundled discount, then, in equilibrium, and relative to the situation without bundling:

(i) the headline prices for the low quality bundling firms will rise to $p_X = p_Y = 0.34815s$.

(ii) the headline prices for the high quality firms will fall to $P_X = P_Y = 0.66374s$.

(iii) the price of the bundle, net of the discount, will fall to $p_X + p_Y - \beta_B s = 0.58494s$.

(iv) consumer surplus will rise to $CS_1 = 2.4405s_B - 0.4405s_A$.

(v) the profit of each of the bundling firms will rise to $\Pi_{BX} = \Pi_{BY} = 0.22433s$ and the profit of each of the non-bundling firms will fall to $\Pi_{AX} = \Pi_{AY} = 0.88110s$.

(vi) welfare will fall to $TW_1 = 1.7704s_A + 0.22964s_B$. \hfill \blacksquare

As the previous Lemma shows, offering a bundled discount to those consumers opting for the acquisition of the low quality bundle will increase the individual profit of the firms offering the discount and, at the same time, will decrease the profit of the high quality producers (not involved in a similar discounting scheme). In addition, even though consumers’ surplus is enhanced, overall, social welfare will decrease after the discount introduction. We defer the discussion of the induced effects on consumer surplus and on social welfare regarding this scenario and also the following ones to Section 3.4.

\textsuperscript{11} Also, some consumers of $B_X, A_Y$ will switch to $A_X, A_Y$ but this does not affect the total demand for $B_Y$.

\textsuperscript{12} Evaluated at the no-discount equilibrium, this internal strategic effect boils down to $\left(\frac{1}{3} s \left( \frac{1}{3} - \frac{1}{3} \right) \frac{5}{6} s + \left( \frac{1}{3} \left( \frac{1}{3} - \frac{1}{3} \right) \right) \frac{5}{6} s = 0. \right.

\textsuperscript{13} With simultaneous price and discount decisions, one would have five independent entities making decisions: four firms setting prices and also an “entity” that maximizes the low quality firms’ joint profit with respect to the discount. In our opinion, this alternative scenario is less plausible than the one analyzed in the text. We mention it, however, in order to help to isolate the impact of pre-commitment to a discount, which is one of the consequences of introducing discounts by a pair of single product independent firms.
3.2.3 Scenario 2: Unilateral bundling by the high quality firms

We now analyze the scenario in which \(A > 0\) while \(B = 0\). In this scenario, making use of Appendix B, one concludes that equilibrium prices for the high quality pair of firms and for the low quality pair of firms are, respectively, given by

\[
\begin{align*}
P_X(\alpha_A, \beta_A) &= P_Y(1 - \alpha_A, \beta_A) = \\
&= \frac{\alpha_A^2 \beta_A^2 (8\beta_A + \beta_A^2 + 8) + \alpha_A \beta_A (32\beta_A - 12\beta_A^2 - 3\beta_A^3 + 32) - 4\beta_A (3\beta_A + 2\beta_A^2 - 4) + 96)}{4 (36 - \alpha_A (1 - \alpha_A) \beta_A^2 - 6\beta_A^4)},
\end{align*}
\]

and

\[
\begin{align*}
p_X(\alpha_A, \beta_A) &= p_Y(1 - \alpha_A, \beta_A) = \\
&= \frac{(\alpha_A^2 \beta_A^2 (\beta_A + 2))^2 + \alpha_A \beta_A (12\beta_A - 12\beta_A^2 - 3\beta_A^3 + 16) + 4 (2\beta_A^3 - 3\beta_A^2 - 4\beta_A + 12)) s}{4 (36 - \alpha_A (1 - \alpha_A) \beta_A^2 - 6\beta_A^4)}.
\end{align*}
\]

As in the previous scenario, the optimal discount level and the discount sharing arrangement is obtained by maximizing the partner firms’ joint profit, in this case \(\Pi_X(\alpha_A, \beta_A) + \Pi_Y(\alpha_A, \beta_A)\), evaluated at the equilibrium prices above, with respect to \(\beta_A\) and \(\alpha_A\). Similarly to what happened in the previous scenario, for any admissible \(\beta_A\), this results in \(\alpha_A^* = 1/2\).

For similar reasons as those indicated with respect to Scenario 1, the high quality headline price increases with \(\beta_A\); the high quality bundle price, net of the corresponding discount, is decreasing in \(\beta_A\); the low quality headline price decreases in \(\beta_A\) and the sum of the headlines prices of two goods of different qualities increases with \(\beta_A\).

The corresponding optimal discount is \(\beta_A^* = 0.28116\), which is also within the assumed range of small discounts. The trade-off faced by the partner high-quality producers when deciding on the optimal level for the discount is qualitatively similar to the one discussed in the previous section for the low quality partner firms. There are, however, the following differences when the effects are evaluated at the no-discount equilibrium:

(a) The initial number of consumers entitled to the discount is four times larger, which results from the fact that the high quality firms have a larger market share;

(b) Each consumer that switches to the high quality components is charged a price which is twice as large as the price paid by consumers who switch to low quality components. The unit cost, however, is the same for both types of quality, and equal to 0;

(c) The effects of the discount on the corresponding quantities are larger (again, twice as large).

Comparison of Figures 3 and 4 illustrates that, indeed, the effect of an increase in the discount on a given firm’s demand is larger in the case of a discount attributed by the high quality producers;
Figure 4: The direct effect on demand of increasing the bundled discount by the high quality producers.

(d) The effect of the discount on the equilibrium headline prices of the bundling partners (positive effect) and of the non bundling firms (negative effect) is larger in the case of bundling by the high quality firms (again, twice as large).

As a result, the overall effect (and also the direct and the strategic effects when taken separately) is four times larger in the scenario under analysis than in the case of bundling by the low quality firms. This explains why the magnitude of the equilibrium discount is larger in the case of unilateral bundling by the high quality firms.

The results are summarized in the following Lemma.

Lemma 4: If only the high quality firms offer the bundled discount, then, in equilibrium, and relative to the situation without bundling:

(i) the headline prices for the low quality firms will fall to $p_X = p_Y = 0.31977s$.
(ii) the headline prices for the high quality bundling firms will rise to $P_X = P_Y = 0.74070s$.
(iii) the price of the bundle, net of the discount, will fall to $P_X + P_Y - \beta^*s = 1.2002s$.
(iv) consumer surplus will fall to $CS_2 = 2.4646s_B - 0.4646s_A$.
(v) the profit of each of the bundling firms will rise to $\Pi_{AX} = \Pi_{AY} = 0.90923s$ and the profit of each of the non-bundling firms will fall to $\Pi_{BX} = \Pi_{BY} = 0.2045s$.
(vi) welfare will fall to $TW_2 = 1.7629s_A + 0.23714s_B$.

Hence, in contrast with the scenario studied in the previous section, consumers’ surplus
will be negatively affected by the introduction of a bundled discount by the high quality producers. As far as producers’ surplus is concerned, the high quality bundling partner firms will benefit from an increase in profits whereas their competitors will see their profits declining. In addition, and despite the fact that aggregate profits increase, total welfare decreases upon the introduction of the discount.

3.2.4 Scenario 3: Bilateral bundling by firms with same quality

We now analyze the case in which $\beta_A$ and $\beta_B$ are both positive. The expressions for the equilibrium prices are those presented in Appendix B. It can be shown that, for any level of the two discounts, if the rival pair of firms sets $\alpha_i = 1/2$, the other pair of firms maximizes its profit by also setting $\alpha_j = 1/2$, where $i, j = A, B$.

Analyzing the equilibrium prices at $\alpha_A = \alpha_B = 1/2$, we conclude that, for all admissible discounts, the high quality and the low quality headline prices increase with the respective discount, the high and low quality bundle prices (net of the corresponding discount) are decreasing with the respective discount, and the sum of the headlines prices of the two different quality products increases with the discounts. The equilibrium discounts are given by $\beta_A^* = 0.33333$ and $\beta_B^* = 0.16666$, which is a corner solution.

The following lemma presents the equilibrium under bilateral bundling.

**Lemma 5:** If both the low and high quality firms offer a bundled discount, then, in equilibrium and relative to the situation without bundling:

(i) the headline prices for the high quality bundling firms will rise to $P_X = P_Y = 0.75455s$.

(ii) the headline prices for the low quality firms will rise to $p_X = p_Y = 0.3465s$.

(iii) the price of the high quality bundle, net of the discount, will decrease to $P_X + P_Y - \beta_A^*s = 1.1758s$ and the price of the low quality bundle, net of the discount, will decrease to $p_X + p_Y - \beta_B^*s = 0.52633s$

(iv) consumer surplus will fall to $CS_3 = 2.4705s_B - 0.47048s_A$

(v) the individual profit of the high quality firms will rise to $\Pi_{AX} = \Pi_{AY} = 0.90047s$ and the individual profit of the low quality firms will fall to $\Pi_{BX} = \Pi_{BY} = 0.20643s$.

(vi) welfare will fall to $TW_3 = 1.7433s_A + 0.25668s_B$.

Hence, in this bilateral bundling scenario, only the high quality pair of firms will be better off with the introduction of the two bundled discounts. In addition, and perhaps most importantly, both total welfare and consumers’ surplus will decrease.
3.3 Bundling by firms with different quality levels

Assume now that one or two pairs of firms with different qualities offer a bundled discount. We assume that the high quality producer of $X$ offers a joint discount with the low quality producer of $Y$, denoted by $\beta_{AB}$. In this case, the percentage of the discount financed by the high quality producer is denoted by $\alpha_{AB}$. If the remaining pair of firms also offers a bundled discount, the corresponding notation regarding the discount level and the percentage of the discount financed by the high quality producer will be $\beta_{BA}$ and $\alpha_{BA}$, respectively.

In this scenario, equilibrium prices result from the individual maximization of the following objective functions:

$$
\Pi_{AX} = P_X (Q_{AX,BY} + Q_{AX,AY} + K_{AX}) - \alpha_{AB}\beta_{AB}sQ_{AX,BY}
$$
$$
\Pi_{AY} = P_Y (Q_{BX,AY} + Q_{AX,AY} + K_{AY}) - \alpha_{BA}\beta_{BA}sQ_{BX,AY}
$$
$$
\Pi_{BX} = p_X (Q_{BX,AY} + Q_{BX,BY} + K_{BX}) - (1 - \alpha_{BA})\beta_{BA}sQ_{BX,AY}
$$
$$
\Pi_{BY} = p_Y (Q_{AX,BY} + Q_{BX,BY} + K_{BY}) - (1 - \alpha_{AB})\beta_{AB}sQ_{AX,BY}
$$

The corresponding analytical expressions regarding the equilibrium prices are presented in Appendix B.

3.3.1 Scenario 4: Unilateral bundling by firms with different quality

In this section, we study the effects induced by bundled discounts given by a pair of producers offering different quality levels, assuming that the remaining pair of firms does not offer a discount. Hence, $\beta_{AB} > 0$ while $\beta_{BA} = 0$. The resulting division of consumers interested in buying both products is as in Figure 5.

Equilibrium prices for the firms that provide the discount are given by

$$
P_X = s \left( \frac{8\beta_{AB} - 12\beta_{AB}^2 - 8\beta_{AB}^3 + \beta_{AB}\alpha_{AB} (32\beta_{AB} - 12\beta_{AB}^2 - 3\beta_{AB}^3 + 16) + \beta_{AB}^2 \alpha_{AB} (8\beta_{AB} + \beta_{AB}^2 + 8) + 96}{4 (\alpha_{AB}^2 \beta_{AB}^2 - \alpha_{AB}\beta_{AB}^2 - 6\beta_{AB}^2 + 36)} \right),
$$

$$
P_Y = s \left( \frac{-2\beta_{AB} (6\beta_{AB}^2 - 22\beta_{AB} + \beta_{AB}^3 - 24) + \beta_{AB}\alpha_{AB} (\beta_{AB}^3 - 4\beta_{AB}^2 - 48\beta_{AB} - 32) + \alpha_{AB}^2 \beta_{AB}^2 (8\beta_{AB} + \beta_{AB}^2 + 4) + 48}{4 (\alpha_{AB}^2 \beta_{AB}^2 - \alpha_{AB}\beta_{AB}^2 - 6\beta_{AB}^2 + 36)} \right),
$$
whereas the equilibrium prices regarding the other two firms are

\[
P_Y = s \left( -2\beta_{AB}^2 (\beta_{AB}^2 + 2) + \beta_{AB}\alpha_{AB} (4\beta_{AB}^2 - 24\beta_{AB} + \beta_{AB}^3 - 16) + \beta_{AB}^2\alpha_{AB}^2 (\beta_{AB} + 2)^2 + 96 \right. \\
\left. \frac{4 (\alpha_{AB}^2\beta_{AB}^2 - \alpha_{AB}\beta_{AB}^2 - 6\beta_{AB}^2 + 36)}{4 (\alpha_{AB}^2\beta_{AB}^2 - \alpha_{AB}\beta_{AB}^2 - 6\beta_{AB}^2 + 36)} \right),
\]

\[
p_X = s \left( 4\beta_{AB} (2\beta_{AB} + 1) (\beta_{AB} - 2) + \alpha_{AB}\beta_{AB} (12\beta_{AB} - 12\beta_{AB}^2 - 3\beta_{AB}^3 + 8) + \beta_{AB}^2\alpha_{AB}^2 (\beta_{AB} + 2)^2 + 48 \right. \\
\left. \frac{\alpha_{AB}^2\beta_{AB}^2 - \alpha_{AB}\beta_{AB}^2 - 6\beta_{AB}^2 + 36}{4 (\alpha_{AB}^2\beta_{AB}^2 - \alpha_{AB}\beta_{AB}^2 - 6\beta_{AB}^2 + 36)} \right).
\]

Now, the optimal discount and the discount sharing arrangement are obtained by maximizing the partner firms’ joint profit, \( \Pi_{AX}(\beta_{AB}, \alpha_{AB}) + \Pi_{BY}(\beta_{AB}, \alpha_{AB}) \), evaluated at the equilibrium prices above, with respect to \( \beta_{AB} \) and \( \alpha_{AB} \), yielding \( \alpha_{AB}^* = 1 \) and \( \beta_{AB}^* = 0.1748 \).

**Lemma 6:** If the high quality producer of \( X \) and the low quality producer of \( Y \) offer a bundled discount, then, in equilibrium and relative to the situation without bundling:

(i) the headline prices for the bundling firms will rise to \( P_X = 0.70487s \) and \( P_Y = 0.35426s \).

(ii) the headline prices for non bundling firms will rise to \( p_X = 0.33588s \) and fall to \( p_Y = 0.64574s \).

(iii) the price of the bundle, net of the discount, will decrease to \( P_X + p_Y - \beta_{AB}^*s = 0.88433s \).

(iv) consumer surplus will rise to \( CS_4 = 2.43308s_B - 0.43297s_A \).

(v) the individual profit of the high quality firms will fall to \( \Pi_{AX} = 0.87323s \) and \( \Pi_{AY} = 0.83395s \) and the individual profit of the low quality firms will rise to \( \Pi_{BX} = 0.22563s \) and \( \Pi_{BY} = 0.251s \).
Two aspects are worth of noting with regards to this scenario: (i) the way the discount is financed; and (ii) the way headline prices of the products outside the bundle change with the introduction of the discount.

As mentioned above, the discount can be interpreted as a unit cost, partially incurred by each partner firm, for the units entitled to the discount (which, in the current scenario, are $Q_{ AX, BY }$ units). Hence, in the scenario under analysis, an increase in $\alpha_{ AB }$ induces an increase in the costs of the high quality producer of $X$ and, at the same time, a decrease in the costs of the low quality producer of $Y$ when selling each unit of the bundle. These effects on the partner firms’ costs will in turn make $P_{ X }$ increase and $p_{ Y }$ decrease, and will also affect the prices of the substitute products produced by the non-partner firms, respectively $p_{ X }$ and $P_{ Y }$, in a similar way but by a smaller magnitude, as these are strategic complements. As a result, both price differences, $P_{ X } - p_{ X }$ and $P_{ Y } - p_{ Y }$ will increase with $\alpha_{ AB }$. It should be noted, however, that the impact of changing $\alpha_{ AB }$ on the profit of the low quality producer of $Y$ is substantially larger than the impact of the same change in $\alpha_{ AB }$ on the profits of the high quality producer of $X$. This happens because $Q_{ AX, BY }$ represents a larger fraction of the sales of the low quality producer of $Y$ than of the sales of the high producer of $X$ (the reason being that the high quality producer will have higher aggregate sales).

As a result, $P_{ Y } - p_{ Y }$ ends up increasing more than $P_{ X } - p_{ X }$, meaning that the $A_{ Y, B X }$ pair becomes relatively more expensive than the $A_{ X, BY }$ bundle. Graphically, this implies that the upward sloping line in Figure 6 shifts both upwards and to the right, but the first effect is stronger, implying that the new upward sloping line is above the initial one. In words, although the increase in $P_{ X } - p_{ X }$ makes some consumers switch away from the high quality producer of $X$, the higher increase in $P_{ Y } - p_{ Y }$ more than compensates for this for those consumers who choose the bundle. Hence, as $\alpha_{ AB }$ increases, so does the demand for the bundle, even as its price increases, resulting in higher profits for the bundling firms.\footnote{This contrasts with the cases of symmetric partner firms, such as Scenarios 1 and 2 above. In those cases, bundling firms differ only if they do not share equally the discount cost. Without loss of generality, assume the case of Scenario 1, i.e., the case where there is bundling by the low quality pair of firms and consider that $\alpha < 1/2$. Then, the low quality producer of $X$ incurs in lower costs associated with the production of $Q_{ B X, BY }$ than does the low quality producer of $Y$. This means that it will set lower unit prices and will, as a result, sell an overall higher number units. An increase in $\alpha_{ B }$ will thus have exactly the same effect as the one described in the previous paragraph. However, when $\alpha_{ B }$ exceeds $1/2$ this effect is reversed because the low quality producer of $X$ is now the firm with higher costs, higher prices and lower overall sales and will, therefore, be more sensitive to an increase in the unit costs associated with the $Q_{ B X, BY }$ units.}
As for the headline prices of the products outside the bundle, this scenario contrasts with the other cases of unilateral bundling. In this case, the price of the low quality product that is not included in the bundle, $B_X$, increases whereas in the other cases, the headline prices of products outside the bundle would always decrease. Recall from the discussion in Scenario 1 that the introduction of the bundle has two opposite signed effects on a firm that is not offering a discount: on the one hand, there is a lower demand for its product but, on the other hand, its direct competitor will face “higher costs” in some of the units it sells. In the current scenario, as the high quality producer of $X$ is paying for the entirety of the discount, the increase in its costs is very large and, therefore, for the low quality producer of $X$, the second effect dominates the first one and, as a result, its equilibrium price will increase.

### 3.3.2 Scenario 5: Bilateral bundling by firms with different quality

We now analyze the case in which $\beta_{AB}$ and $\beta_{BA}$ are both positive. The expressions for the equilibrium prices are those presented in Appendix B.

In what follows, we look for a symmetric equilibrium. It can be shown that $\partial (\Pi_A + \Pi_B) / \partial \alpha_{AB}$ and $\partial (\Pi_B + \Pi_A) / \partial \alpha_{BA}$ are both positive for all admissible symmetric discounts and profit sharing rules, i.e. for all $\beta_{AB} = \beta_{BA}$ and $\alpha_{AB} = \alpha_{BA}$. Hence, for any level of the two discounts, we have $\alpha_{AB}^* = \alpha_{BA}^* = 1$. Solving the first-order condition with respect to the discount level and looking for a symmetric solution yields $\beta_{AB}^* = \beta_{BA}^* = 0.22842$. 

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Figure 6: The effect on demand of an increase in $\alpha_{AB}$. 

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[Diagram image]
Lemma 7: If the two pairs of firms of different quality levels offer a bundled discount, then, in equilibrium and relative to the situation without bundling:

(i) the headline prices for the high quality bundling firms will rise to $P_X = P_Y = 0.695\,99\,s$.
(ii) the headline prices for the low quality bundling firms will rise to $p_X = p_Y = 0.367\,6\,s$.
(iii) the price of the bundles, net of the discount, will decrease to $P_X + p_Y - \beta_{AB}s = 0.835\,17\,s$.
(iv) consumer surplus will rise to $CS_{5} = 2.423\,33s_B - 0.423\,31s_A$.
(v) the individual profit of the high quality firms will fall to $\Pi_{A_X} = \Pi_{A_Y} = 0.789\,66\,s$ and the individual profit of the low quality firms will rise to $\Pi_{B_X} = \Pi_{B_Y} = 0.270\,26\,s$.
(vi) welfare will fall to $TW_{5} = 1.696\,5s_A + 0.303\,46s_B$.

Hence, the negotiated bundled discount leads partner firms to raise (all) headline prices relative to the benchmark case in which the products are marketed independently. In addition, in this “crossed” bilateral bundling scenario, the discount associated cost will be fully incurred by the high quality producer in each alliance, implying that high quality firms’ profit will decrease whereas the individual profit of low quality producers will rise with the introduction of the two bundled discounts. Overall, however, total welfare will decrease relative to the no discount benchmark.

3.4 Discussion of the results

In this section we briefly discuss and compare the scenarios above, especially the implications on consumer surplus and total welfare.

The discount introduction will affect different groups of consumers in different ways. Consider first those consumers whose choices do not change with the discount introduction. Within this group of consumers, there are different cases to consider that we discuss in turn. Those consumers that purchase the bundle which now entitles them to the discount will be better off as the bundle price decreases. Consumers who do not purchase any of the products in the bundle will also benefit from a price decrease. However, consumers who purchase only one of the products in the bundle will see the price of that product increase and the price of the other one increase or decrease but will be, in any case, worse-off. Additionally, those consumers who only value one of the products will see the corresponding headline price increase or decrease, depending on whether it is present in the bundle or not. To complicate things further, there exist also some consumers who will make different choices after the bundle is introduced and may pay a higher price for more quality or a lower price for lower quality than in the benchmark case. Some of these consumers will be better off while others will be worse off.
If one ranks the scenarios in terms of consumer surplus, the worst case is Scenario 3, followed by Scenario 2, both leading to a decrease in consumer surplus. Scenarios 1, 4 and 5 all lead to increases in consumer welfare, with the latter resulting in the highest increase in consumer surplus. It should be highlighted that in all scenarios where the pair of high quality firms offers a bundled discount, consumers’ surplus decreases, whereas if this pair does not offer a discount then consumer surplus will increase. In particular, in the case of bundling by the high quality firms, the price effects are stronger and the increase in the headline price of the high quality price is very large. This will hurt those consumers who will keep on purchasing the unbundled high quality product to such an extent that will more than compensate for any benefits accruing to other consumers.

As for total welfare, it decreases in all scenarios where bundled discounts are introduced. Notice that in our setting, welfare depends only on the quality of products purchased by consumers since: (i) prices and discounts are mere transfers between consumers; and (ii) quality is valued by all consumers and produced with no additional costs. Hence, any change that makes consumers switch to pairs of products with more quality (such as switching from $B_X, B_Y$ to $A_X, B_Y$ or from $A_X, B_Y$ to $A_X, A_Y$) is welfare enhancing. Indeed, if one ranks scenarios 0, 1, 4 and 5, from the highest to the lowest in terms of welfare, one gets the exactly the same ranking that would be obtained when ordering those scenarios in terms of the total demand for high quality units, as illustrated in Table 1. Nonetheless, the total demand faced by the high quality firms, presented in the last row of Table 1, is not a sufficient statistic for welfare, because quality is not valued equally by all consumers. In fact, scenarios 2 and 3 represent a decrease in welfare, despite the fact that, in both cases and relative to the no discount benchmark, there is an increase in the aggregate number of high quality units sold. Take, for instance, scenario 2. The introduction of the high quality bundle will decrease the bundle net price but, by substantially hiking the high quality headline price and lowering the low quality headline price, it will actually lead to a decrease in total welfare. Some of the consumers who are interested in buying both products will switch to pairs of products of higher quality but others, as well as some of the consumers who only value one of the goods, will switch from the high to the low quality version of the product they care for. Initially, the indifferent consumer between the high and low quality versions of a given good takes the same value both for product $X$ and product $Y$. In particular, for all types of consumers, the indifferent consumer has a valuation for quality of $\theta_j^k = 1/3$. Consider now the following changes in prices which result from the introduction of a bundled discount by the high quality pair of firms: the price of the high quality bundle decreases, the headline prices of the high
quality producers increase and the headline prices of the low quality producers decrease. As a result, the indifferent consumers will change in opposite directions. More consumers who care for both products will purchase higher quality products, but more consumers who care for only one of the available products will purchase lower quality products. The aggregate number of high quality sales increases but welfare decreases because the consumers who will switch from the low to the high quality products have a lower valuation for quality than those consumers who will switch from the high to the low quality version of a given product. As for scenario 3, it fares worse than Scenario 2 because the introduction of the low quality discount will make less consumers purchase the high quality products.

The overall ranking, in terms of welfare of the scenarios is: 0, 1, 2, 4, 3, 5, where Scenario 0 corresponds to the highest welfare whereas Scenario 5 is the one in which welfare is the lowest. The cases of bundling involving different quality firms are, therefore, amongst those that lead to higher welfare losses. This happens because these are the discount schemes that have a larger impact in terms of reducing the sales of the high quality pair, $Q_{AX,AY}$, as Table 1 illustrates. Relatedly, notice that inspection of Figure 1 shows that the number of consumers indifferent between the two pairs of products with different quality and the high quality pair is very large. This then implies that discounts introduced by firms with different qualities, either unilateral or bilateral, will induce a substantially reduction in the number of individuals who purchase the high quality pair, thereby reducing welfare.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{AX,AY}$</td>
<td>0.44444</td>
<td>0.46222</td>
<td>0.70046</td>
<td>0.73116</td>
<td>0.33678</td>
<td>0.19641</td>
</tr>
<tr>
<td>$Q_{AX,BY}$</td>
<td>0.22222</td>
<td>0.18085</td>
<td>0.08094</td>
<td>0.03177</td>
<td>0.36045</td>
<td>0.39680</td>
</tr>
<tr>
<td>$Q_{BX,AY}$</td>
<td>0.22222</td>
<td>0.18085</td>
<td>0.08094</td>
<td>0.03177</td>
<td>0.24616</td>
<td>0.39680</td>
</tr>
<tr>
<td>$Q_{BX,BY}$</td>
<td>0.11111</td>
<td>0.17609</td>
<td>0.13766</td>
<td>0.20529</td>
<td>0.05660</td>
<td>0.00100</td>
</tr>
<tr>
<td>$K_{AX} + K_{AY}$</td>
<td>1.33333</td>
<td>1.36888</td>
<td>1.15813</td>
<td>1.18395</td>
<td>1.33925</td>
<td>1.34322</td>
</tr>
<tr>
<td>$K_{BX} + K_{BY}$</td>
<td>0.66667</td>
<td>0.63118</td>
<td>0.84186</td>
<td>0.81610</td>
<td>0.66046</td>
<td>0.65680</td>
</tr>
<tr>
<td>High quality</td>
<td>2.66666</td>
<td>2.65449</td>
<td>2.72098</td>
<td>2.70987</td>
<td>2.61977</td>
<td>2.52968</td>
</tr>
</tbody>
</table>

Table 1: Consumer shares in each scenario.

4 Coalition formation stage

In this section, we discuss the simultaneous decisions by the four firms in the industry regarding whether or not to participate in a discount scheme. In particular, we assume that
firms choose simultaneously between one of three following alternative strategies: (i) do not offer any discount; (ii) offer a joint discount with a high quality producer; and (iii) offer a joint discount with a low quality producer. We denote these alternatives by \(N\), \(A\) and \(B\), respectively. Let \(S_{Ax} = \{N, A, B\}\) denote the set of alternative strategies of the high quality producer of \(X\) and likewise for the other three firms. Let \(s_{Ax} \in S_{Ax}\) denote the strategy chosen by firm \(A_X\) and let \(S = (s_{Ax}, s_{Ay}, s_{Bx}, s_{By})\) denote a profile of strategies. In total, there are 81 such profiles.

Discounts will only be introduced when at least two firms have coincident matching intentions. Clearly, \(S = (A, A, B, B)\) leads to Scenario 3, \(S = (B, B, A, A)\) leads to Scenario 5, the eight cases of the type \(S = (A, A, s_{Bx}, s_{By})\), with \((s_{Bx}, s_{By}) \neq (B, B)\), all lead to Scenario 2, the eight cases of the type \(S = (s_{Ax}, s_{Ay}, B, B)\), with \((s_{Ax}, s_{Ay}) \neq (A, A)\), all lead to Scenario 1, the eight cases of the type \(S = (B, s_{Ay}, s_{Bx}, A)\), with \((s_{Ay}, s_{Bx}) \neq (B, A)\), and the eight cases of the type \(S = (s_{Ax}, B, A, s_{By})\), with \((s_{Ax}, s_{By}) \neq (B, A)\), all lead to Scenario 4, and all the remaining cases lead to Scenario 0.

**Proposition 1:** Bilateral bundling by firms with the same quality is the unique Strong Nash Equilibrium of the game.

Scenarios 4 and 5 cannot be a Nash equilibrium of this game. If one of the high quality firms involved in a discount with a low quality producer deviates, for instance by choosing \(N\), its payoff will increase both when the other pair of firms is offering a discount or not (See Lemma 6 and Lemma 7).\(^{15}\)

All other scenarios can arise in equilibrium because a mutual agreement is needed to introduce a discount. For instance, \(S = (N, N, N, N)\) is a Nash equilibrium because no firm can move away from Scenario 0 by unilaterally deviating to another strategy. However, it is not a Strong Nash Equilibrium, because any pair of producers of the same quality can cooperatively deviate and introduce a discount, thus benefitting both firms. The same happens when the strategy profile is such that it leads to Scenarios 1 or 2: the pair of mismatched firms would profitably cooperatively deviate and offer a bundled discount. Hence, the case of bilateral bundling by firms with the same quality stands out as the unique Strong Nash Equilibrium of the game. In fact, none of the high quality firms can be induced to cooperatively deviate from this equilibrium because this results in the highest payoff each of these firms can obtain. As

\(^{15}\)In the case of unilateral bundling, the high quality firm producing \(i\) will increase its payoff from \(\Pi_{Ai} = 0.87323s\) in Scenario 4 to \(\Pi_{Ai} = 0.8888s\) in Scenario 0 by deciding not to offer a discount with a low quality firm. In the case of bilateral bundling, the high quality firm will increase its payoff from \(\Pi_{Ai} = 0.78966s\) in Scenario 5 to \(\Pi_{Ai} = 0.83395s\) in Scenario 4.
for the low quality firms, any deviation that does not include a high quality firm will lead to scenario 2, where their profit is smaller.

It is important to highlight that this bilateral bundling scenario where partner firms offer products of similar quality is the one leading to the most adverse consequences in terms of consumer welfare and to the second worse consequences in terms of social welfare. This then suggests that competition authorities should be more vigilant with regards to bundling discounting by independent producers of vertically differentiated goods. In the next section, we discuss the robustness of this result to the adoption of different coalition formation games.

5 Robustness

In this section we discuss the robustness of our results to different assumptions regarding both the competition stage and the coalition formation stage.

5.1 Competition stage

One important result is that in the case of bilateral bundling by firms offering products of the same quality consumer surplus will decrease. We now show that this result does not depend critically on three of our assumptions, namely, on the existence of consumers who care only for one of the available products, on the definition of small discounts and on the assumption of small discounts.

Consider first the case in which there are no consumers interested in only one of the products and assume that both pairs of firms offering goods of the same quality offer (small) bundled discounts which they fund equally, \( \alpha_A = \alpha_B = 1/2 \). Then, it is straightforward to show that the equilibrium prices are given by

\[
P_X = P_Y = \frac{(12\beta_A + 6\beta_B + 5\beta_A\beta_B + 6\beta^2_A + \beta^3_B + \beta^3_B + 4\beta_A\beta^2_B + 4\beta^2_A\beta_B + 8)s}{2(5\beta_A + 5\beta_B + 6)}
\]

\[
p_X = p_Y = \frac{(2\beta_A + 6\beta_B + 5\beta_A\beta_B + \beta^3_A + 6\beta^2_B + \beta^3_B + 4\beta_A\beta^2_B + 4\beta^2_A\beta_B + 4)s}{2(5\beta_A + 5\beta_B + 6)}
\]

Simultaneous maximization by each pair of firms of their corresponding joint profits yields \( \beta_A = 0.33333 \) and \( \beta_B = 7.931 4 \times 10^{-2} \) (again, a corner solution in the case of \( \beta_A \)), from where one obtains the equilibrium headline prices \( P_X = P_Y = 0.828 20s \) and \( p_X = p_Y = 0.334 46s \).

Table 2 illustrates the impact of the introduction of these discounts on the different groups of consumers. In particular, it presents the ex-ante and ex-post consumer choices, the normalized changes in price faced by each consumer group, and the effect on quality and on the aggregate
consumer surplus for each group of consumers. As this Table demonstrates, consumers whose surplus decreases with the introduction of the discounts are those who, before the discount introduction, were buying a pair of products with different quality. This is true independently of whether or not these consumers decide to change their consumption pattern after the discounts introduction. In addition, the welfare loss faced by these consumers more than compensates for any reduction in the bundle prices (and the associated welfare gain) of the remaining groups of consumers, that were buying a pair of products of the same quality before the discounts introduction and keep on buying a bundle of products of the same quality after the discounts are introduced, thereby benefiting from a decrease in the price paid for the chosen bundle. As a result, overall consumers’ surplus decreases in this bilateral bundling scenario, confirming the qualitatively result of the model presented in the previous sections.

<table>
<thead>
<tr>
<th>ex-ante</th>
<th>ex-post</th>
<th>% consumers</th>
<th>$\Delta P \times \frac{100}{s}$</th>
<th>$\Delta Quality \omega$</th>
<th>$\Delta CS \times \frac{100}{s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_X, A_Y$</td>
<td>$B_X, A_Y$</td>
<td>6.848 3</td>
<td>+16.266</td>
<td>0</td>
<td>−1.1139</td>
</tr>
<tr>
<td>$A_X, A_Y$</td>
<td>$A_X, A_Y$</td>
<td>44.221</td>
<td>−1.0267</td>
<td>0</td>
<td>+0.45417</td>
</tr>
<tr>
<td>$A_X, B_Y$</td>
<td>$A_X, B_Y$</td>
<td>6.848 3</td>
<td>+16.266</td>
<td>0</td>
<td>−1.1139</td>
</tr>
<tr>
<td>$B_X, B_Y$</td>
<td>$B_X, B_Y$</td>
<td>11.111</td>
<td>−7.7061</td>
<td>0</td>
<td>+0.85623</td>
</tr>
<tr>
<td>$B_X, A_Y$ or $A_X, B_Y$</td>
<td>$A_X, A_Y$</td>
<td>2 × 8.878 3</td>
<td>+32.307</td>
<td>$\theta_X$ or $\theta_Y$</td>
<td>$2 \times (-0.63342)$</td>
</tr>
<tr>
<td>$B_X, A_Y$ or $A_X, B_Y$</td>
<td>$B_X, B_Y$</td>
<td>2 × 6.495 3</td>
<td>−41.039</td>
<td>$-\theta_X$ or $-\theta_Y$</td>
<td>$2 \times (-0.18495)$</td>
</tr>
<tr>
<td>$A_X, A_Y$</td>
<td>$B_X, B_Y$</td>
<td>0.2230 2</td>
<td>−74.373</td>
<td>$-\theta_X - \theta_Y$</td>
<td>0.0072561</td>
</tr>
</tbody>
</table>

Table 2: Effect on consumers of bilateral bundling by producers with the same quality.

We now turn to the definition of small discounts. We present an alternative definition of small discounts and characterize the outcome of scenario 3 when this definition is used. Instead of being based on the average price of the products in the bundle, in this alternative definition an upper bound for the admissible discounts is established for each scenario. In particular, in what follows we define, for each scenario, a small discount as a discount that leads to an equilibrium in the pricing stage such that the four possible combinations of products have a positive equilibrium market share for all possible ways of sharing the discounts and for all discount levels of the competitors (if any). This allows us to define an upper bound, common to all firms, below which discounts are admissible. Table 3 presents the upper bound for the two definitions of small discounts, as well as the corresponding equilibrium discounts.
As can easily be seen this definition of the discount is less restrictive, with the exception of the high quality firms in Scenario 3.\(^{16}\) The corresponding equilibrium discounts in Scenario 3 would then be \(\beta_A^* = 0.31476\) and \(\beta_B^* = 0.20435\) and, under this alternative definition, Lemma 5 would have to be restated as follows:

**Lemma 5\(^\prime\):** If both the low and high quality firms offer a bundled discount, then, in equilibrium and relative to the situation without bundling:

(i) the headline prices for the high quality bundling firms will rise to \(P_X = P_Y = 0.74712s\).

(ii) the headline prices for the low quality firms will rise to \(p_X = p_Y = 0.35476s\).

(iii) the price of the high quality bundle, net of the discount, will decrease to \(P_X + P_Y - \beta_A^*s = 1.1833s\) and the price of the low quality bundle, net of the discount, will decrease to \(p_X + p_Y - \beta_B^*s = 0.50517s\).

(iv) consumer surplus will fall to \(2.4692s_B - 0.46916s_A\)

(v) the individual profit of the high quality firms will rise to \(\Pi_X = \Pi_Y = 0.89918s\) and the individual profit of the low quality firms will fall to \(\pi_X = \pi_Y = 0.20726s\).

(vi) welfare will fall to \(0.25632s_B + 1.7437s_A\).

Note also that, under this alternative definition of small discounts, the other lemmas would suffer no change and all results would hold qualitatively.\(^{17}\)

\(^{16}\)In our initial definition of small discounts the constraint is binding for the two pairs of firms. In this alternative definition, it is only binding for the high quality pair of firms and if the number of consumers valuing only one the products was large enough, no constraint would be binding in Scenario 3.

\(^{17}\)If one considered from the outset that \(\alpha_A = \alpha_B = 1/2\) whenever the firms involved in the discount were symmetric, the constraint regarding the admissible discount levels would be further relaxed. In that case, the equilibrium discounts in Scenario 3, the only one affected, would be \(\beta_A^* = 0.37284\) and \(\beta_B^* = 0.26347\), and all the results above (as well as the comparisons between scenarios) would still qualitatively hold.
Finally, let us consider now the case of “large” discounts by both pairs of firms offering products of similar quality level and, for the sake of tractability, let all consumers have a sufficiently large valuation for the two products so that every single consumer is interested in purchasing both available products. If discounts are so large that all consumers opt for buying one of the available bundles, then a given consumer will choose the low quality bundle if and only if

\[ \theta_X s_B + \theta_Y s_B - p_X - p_Y + \beta_B s > \theta_X s_A + \theta_Y s_A - P_X - P_Y + \beta_A s \]

or \( \theta_Y < \frac{p_X + p_Y - \beta_A s}{s} - \frac{p_X + p_Y - \beta_B s}{s} - \theta_X \). This means that the demand for the low quality bundle is given by \( Q_{B_X,B_Y} = \frac{1}{2} \left( \frac{p_X + p_Y - \beta_A s}{s} - \frac{p_X + p_Y - \beta_B s}{s} \right)^2 \), provided that \( 0 < \frac{p_X + p_Y - \beta_A s}{s} - \frac{p_X + p_Y - \beta_B s}{s} < 1 \). The equilibrium prices will then be the solution to

\[
\frac{\partial \left( (P_i - \frac{1}{2} \beta_A s) (1 - Q_{B_X,B_Y}) \right)}{\partial P_i} = \frac{\partial \left( (p_i - \frac{1}{2} \beta_B s) Q_{B_X,B_Y} \right)}{\partial p_i} = 0
\]

with \( i = X, Y \), from where the following symmetric solution is obtained: \( P_X = P_Y = \frac{1}{2} s \beta_A + \frac{1}{3} s \sqrt{6} \) and \( p_X = p_Y = \frac{1}{2} s \beta_B + \frac{1}{6} s \sqrt{6} \) with \( Q_{B_X,B_Y} = 1/3 \). Consumer surplus in this case is given by \( CS = \frac{1}{27} (17 \sqrt{6} s_B + s_A (27 - 17 \sqrt{6})) \) which is always lower than in the no-discounts benchmark scenario, which again confirms the qualitative result of the model presented in the previous sections. In Gans and King (2006) the equilibrium prices, net of the discount, are exactly the same as in the no-discounting benchmark. The distribution of consumers between products is, however, different and, since this new distribution is associated with higher transportation costs, consumers’ welfare ends up being reduced. Their model’s symmetry assumptions imply that firms have exactly the same demand both before and after the discounts are introduced and, therefore, have the same profit. With vertical differentiation, however, there is no such symmetry. To understand the differences, consider those consumers who, in the absence of bundled discounts, choose the low quality version of product X and the high quality version of product Y.\(^{18}\) Assume now that the headline prices increase significantly, inducing consumers not to buy unbundled products but that discounts are such that the prices net of the discount remain unchanged, as happens in the equilibrium studied by Gans and King (2006). When deciding between the high quality and the low quality bundles, the above mentioned consumers will select the high quality bundle if and only if \( \theta_X + \theta_Y > \frac{2}{3} \).\(^{19}\) Given our assumption of consumers being uniformly distributed, the number of consumers that will

\[^{18}\text{These consumers are those with } (\theta_X, \theta_Y) \in \left[ 0, \frac{1}{2} \right] \times \left[ \frac{1}{3}, 1 \right].\]

\[^{19}\text{Moving from } B_X A_Y \text{ to } A_X A_Y \text{ represents an additional quality valuation of } \theta_X s \text{ and an increase in price of } \frac{1}{2} s. \text{ Moving form } B_X A_Y \text{ to } B_Y B_X \text{ represents a decrease in quality valued at } -\theta_Y s \text{ but also a decrease in price of } \frac{1}{2} s. \text{ Hence, consumers such that } \theta_X s - \theta_Y s + \frac{1}{2} s \Leftrightarrow \theta_X + \theta_Y > \frac{2}{3} \text{ will change to the high quality bundle and the remainder will change to the low quality bundle.}\]
change to the high quality bundle is three times larger than the number of consumers that will change to the low quality bundle. This means that, at the original (net) prices and after this redistribution of consumers, the number of inframarginal consumers is very large for the high quality firms. Hence, the high quality firms have a strong incentive to hike their prices net of the discount above the initial equilibrium level and, in equilibrium, are followed by the low quality firms.

5.2 Coalition formation stage

In this Section, we consider several alternative approaches to the coalition formation stage. As a first alternative, we use an adaptation of Horn and Persson’s (2001) dominance relation that was proposed originally to identify the outcome of an endogenous merger game. According to Horn and Persson (2001), if an ownership structure (in the case of mergers) is ‘dominated’ by another one, the former will not be the outcome the game. Hence, an equilibrium coalition structure must be undominated when compared to all possible alternatives. In addition, one coalition structure is said to dominate another one if the (aggregate) payoff of each decisive group, i.e., the set of players who “have the power of enforcing one structure over the other”, is larger under the dominant structure. The idea is then that if the players who can enforce coalition structure 1 over coalition structure 2 have a higher aggregate profit in structure 1, then they will find a way to share this aggregate profit in such a way that is beneficial for both and, hence, will enforce it. As a result, structure 2 cannot be the outcome of the coalition formation game. This reasoning implies, naturally, the ability to share aggregate profits in a way that all decisive owners are better off, which, in our circumstances might involve side payments. Note, however, that we can easily establish that all scenarios but Scenario 3 (bilateral bundling by firms of the same quality) are dominated without the need to resort to side payments. For instance, unilateral bundling by any pair of firms with the same quality is dominated by bilateral bundling by firms with the same quality. This happens because the individual profit of each decisive owner is larger with bilateral bundling (i.e., there would be no need of a side payment to enforce bilateral bundling over unilateral bundling by firms with the same quality level). As for the no bundling scenario, it is also dominated, without the need to resort to side payments, by any form of unilateral bundling by firms offering goods of the same quality. Finally, we discuss the case of bundling by firms offering goods of different quality. In the absence of side payments, any high quality firm will gain by unilaterally breaking away from the discounting scheme. The gains would then be $0.88889s - 0.87323s > 0$ in the case of a deviation from the unilateral bundling scenario to the no-discount benchmark case or $0.83395s - 0.78966s > 0$ in
the case of a deviation from the (crossed) bilateral bundling scenario to the unilateral bundling involving a pair of firms offering goods of different quality.\textsuperscript{20}

A second possible alternative would be to consider that firms play the following sequential coalition formation game.\textsuperscript{21} Let nature choose randomly one of the firms. This firm is allowed to offer (or not) a firm selling the other product the opportunity of jointly introducing a bundled discount. If the offer is accepted, these firms leave the game and nature chooses another firm to make a similar offer. If it is rejected, nature chooses another firm of the remaining three to make the offer and so forth until all firms have been matched or have had the opportunity of making an offer. We assume that a firm that has rejected an offer will not make an offer to the same firm whose offer it had previously rejected. If a high quality firm, at any stage of the game, accepts an offer from a firm with the same quality, it will make a profit of at least $0.90047s$, which occurs if the low quality pair also ends up offering a discount. If it rejects it, it will get at most $0.88889s$, which occurs if no discount ends up being offered. Hence, any high quality firm will always accept an offer from another high quality firm. Therefore, the first high quality firm to move will make an offer to the other high quality firm, ensuring at least $0.90047s$. In case the high quality firm made instead an offer to a low quality firm or if it decided to make no offer, it would at most gain $0.88889s$ (corresponding to the case where no offer had been accepted). But, anticipating this, if any high quality firm receives an offer from a low quality firm before the first high quality firm gets the chance to make one, it will reject it. Therefore, if a low quality firm moves first, it will make an offer to the other low quality firm which will be accepted and, whenever a high quality firm moves, it will make an offer to the other high quality firm which will also be accepted. As a result, the outcome of this sequential coalition formation game is, again, Scenario 3.

Finally, as a third alternative, we assume that only firms offering the same quality level might be able to offer bundled discounts together.\textsuperscript{22} In particular, we assume that the first

\textsuperscript{20}Note that if one allowed for side payments, the low quality firm might pay the high quality firm not to break away, but could not match the highest offer that the other high quality firm could make to its former partner.

\textsuperscript{21}This game is inspired on a sequential merger formation game used by Vasconcelos (2011) to study merger patterns in a setting where mergers are motivated by prospective efficiency gains and must be previously submitted to an Antitrust Authority for approval.

\textsuperscript{22}The motivation for this assumption is that a producer offering a high quality product is probably not interested in being allied with a low quality producer (of the other good) as this may seriously hurt the reputation of its firm. The pharmaceuticals industry also provides a compelling example of a situation where high quality firms, at least as perceived by consumers, may choose not to be confused with lower quality firms. In fact, there are examples of producers of branded pharmaceuticals creating another firm to produce the corresponding generic. We analyze this alternative so that our results are directly compared to the literature.
stage of our proposed game, the coalition formation stage, is as follows. Each pair of firms offering the same quality level is asked whether or not the involved firms want to jointly offer a discount. Each pair of firms may then either answer yes, “Y”, or no, “N”. After the “coalitions” have been formed, the game moves to the competition stage where discounts are set first and prices are set subsequently. The payoff matrix corresponding to the firms playing this coalition game is as follows, where indicated payoffs correspond to the individual payoff per member of the coalition.

<table>
<thead>
<tr>
<th></th>
<th>High quality firms</th>
<th>Low quality firms</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>Y</td>
<td>0.20643s, 0.90047s</td>
<td>0.22433s, 0.88110s</td>
</tr>
<tr>
<td>N</td>
<td>0.20450s, 0.90923s</td>
<td>0.22222s, 0.88889s</td>
</tr>
</tbody>
</table>

Clearly, in this restricted coalition formation game, offering a bundled discount is a dominant strategy both for the high quality firms and for the low quality firms. Hence, the Nash equilibrium of this game corresponds to the scenario of bilateral bundling, as in Gans and King (2006). However, and in contrast with Matutes and Regibeau (1992), Gans and King (2006), Thanassoulis (2007) or Maruyama and Minamikawa (2009), in this bilateral bundling scenario the high quality firms earn higher profits than in the status quo no-discounting situation. So, in our setting, firms do not find themselves in a prisoners’ dilemma: high quality firms have very strong incentives to participate in bilateral bundling.23

6 Conclusion

Bundled discounts provide purchasers the opportunity to pay less for a bundle of products than if they purchased each item in the package separately at the corresponding headline price. Despite the fact that this business practice is ubiquitous in today’s society, economic theory has devoted very scarce attention to this issue until recently.

The present paper studies the consequences of bundled discounts in an oligopoly setting where pairs of firms sell vertically differentiated and otherwise unrelated products. More specifically, we investigate the effects induced by the introduction of bundled discounts under five

\[^{23}\text{Strictly speaking, Gans and King (2006) establish an invariance result, i.e. equilibrium profits with discounts coincide with what they were without bundling. However, if one considered the existence of a small fixed cost of introducing a discount, this would result in a prisoner’s dilemma.}\]
different scenarios: 
(i) unilateral bundled discount by the low quality firms; 
(ii) unilateral bundled discount by the high quality firms; 
(iii) bilateral bundling by firms with similar quality levels; 
(iv) unilateral bundled discount by a high and a low quality firm; and 
(v) bilateral bundling by firms offering goods of different quality level.

Some interesting results are obtained regarding the competitive effects of bundled discounts, allowing to shed some light on the understanding of the potential antitrust risks associated with this particular type of discount arrangements. First, whenever bundled discounts are offered by (one or two pairs of) firms, then, relative to the no-discounting benchmark case, the headline prices of the bundling firms always increase whereas the headline prices of the firms not involved in discounting (if any) decrease in most cases. Second, in none of the five studied scenarios should bundled discounts be free of antitrust concerns: bundled discounts induce a decrease in consumer surplus in two cases and a decrease in total welfare in all five cases. Third, when firms make simultaneous or sequential decisions regarding their eventual participation in a bundled discount scheme, the robust outcome is that of bilateral bundling by firms with similar quality and the introduction of the discounts is, in equilibrium, only profitable for some firms. In addition, consumer surplus is at the lowest level of all cases considered.

Armstrong (2010) highlights that “when firms offer independent products (i.e. the purchase of one product has no impact on a consumer’s willingness to pay for a second product), Pareto improvements are possible if firms coordinate their pricing policies. This is achieved by means of bundling so that a consumer is offered a discount if she buys both products. Such bundling discounts, if they can be implemented by separate firms, can improve both profits and consumer surplus.” Our contribution to this discussion is the identification of a context – bundling by vertically differentiated competitors – in which, while profitable for some firms, bundling discounts by independent firms do harm consumers. Hence, our results differ from the previous literature on bundling discounts by independent firms in two important ways. First, they embody different predictions regarding the induced impact of the adoption of this discounting schemes on both consumers’ and total welfare. Second, the typical prisoner’s dilemma identified in most of the existing literature is absent in our setting, meaning that the practice of mixed bundling is shown to be profitable for some firms. This then suggests that competition authorities should scrutinize in detail the use of bundled discounts in industries where independent suppliers offer vertically differentiated products.
Appendix A - Proofs

Proof of Lemma 1: Assume initially that there are no discounts. Consumers purchase $i = X, Y$ from $A_i$ if and only if
\[ V + \theta_i s_A - P_i > V + \theta_i s_B - p_i \iff \theta_i > \theta_i^*: = \frac{P_i - p_i}{s} \]

Assume now that the high quality firms introduce a discount $\beta_A$. Then:

Consumers with $\theta_X > \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y > \theta_Y^* = \frac{P_Y - p_Y}{s}$ will still purchase $A_X, A_Y$.

Consumers with $\theta_X > \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y < \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase $A_X, A_Y$ if

$\theta_X s_A - P_X + \theta_Y s_A - P_Y + s\beta_A > \theta_X s_A - P_X + \theta_Y s_B - p_Y \iff \theta_Y > \theta_Y^{AA} := \frac{P_Y - p_Y - s\beta_A}{s}$.

Assume now that the low quality firms also introduce a discount, $\beta_B$. Then:

Consumers with $\theta_X < \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y < \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase $A_X, A_Y$ if

$\theta_X s_A - P_X + \theta_Y s_A - P_Y + s\beta_A > \theta_X s_B - p_X + \theta_Y s_B - p_Y \iff \theta_Y > \theta_Y^{SB} := \frac{P_Y - p_Y - s\beta_A}{s}$.

Consumers with $\theta_X < \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y > \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase $A_X, A_Y$ if

$\theta_X s_A - P_X + \theta_Y s_A - P_Y + s\beta_A > \theta_X s_B - p_X + \theta_Y s_B - p_Y \iff \theta_Y > \theta_Y^{BB} := \frac{P_Y - p_Y + s\beta_B}{s}$.

Assume now that the low quality firms also introduce a discount, $\beta_B$. Then:

Consumers with $\theta_X < \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y < \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase $A_X, A_Y$ if

$\theta_X s_B - p_X + \theta_Y s_B - p_Y + s\beta_B > \theta_X s_B - p_X + \theta_Y s_A - P_Y \iff \theta_X < \theta_X^{AA} := \frac{P_X - p_X + p_Y + s\beta_A - \theta_Y}{s}$.

Assume now that the low quality firms also introduce a discount, $\beta_B$. Then:

Consumers with $\theta_X > \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y > \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase $B_X, B_Y$ if

$\theta_X s_B - p_X + \theta_Y s_B - p_Y + s\beta_B > \theta_X s_B - p_X + \theta_Y s_B - p_Y \iff \theta_X < \theta_X^{BB} := \frac{P_X - p_X + p_Y + s\beta_B - \theta_Y}{s}$.

Assume now that the low quality firms also introduce a discount, $\beta_B$. Then:

Consumers with $\theta_X < \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y > \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase $B_X, B_Y$ if

$\theta_X s_B - p_X + \theta_Y s_B - p_Y + s\beta_B > \theta_X s_B - p_X + \theta_Y s_A - P_Y \iff \theta_X < \theta_X^{AA} := \frac{P_X - p_X + p_Y + s\beta_A - \theta_Y}{s}$.

Assume now that the low quality firms also introduce a discount, $\beta_B$. Then:

Consumers with $\theta_X > \theta_X^* = \frac{P_X - p_X}{s}$ and $\theta_Y > \theta_Y^* = \frac{P_Y - p_Y}{s}$ will purchase $B_X, B_Y$ if

$\theta_X s_B - p_X + \theta_Y s_B - p_Y + s\beta_B > \theta_X s_B - p_X + \theta_Y s_A - P_Y + s\beta_A$.

which is equivalent to

$\theta_X < \frac{P_X - p_X + p_Y - p_Y + s\beta_B - s\beta_A - \theta_Y}{s} = \theta_X^{AA} + \theta_Y^{BB} - \theta_Y$.

The demand functions result from calculating the relevant areas in the $(\theta_X, \theta_Y)$-space with $(\theta_X, \theta_Y) \in [0, 1]^2$. 

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The same type of proof applies to the case of bundling by firms with different quality levels.

Sketch of the proofs of Lemmas 2 to 6.

Using the demand functions presented in Lemma 1 we solved, for each scenario, the system of first-order-conditions to obtain the equilibrium prices as a function of: (i) the discounts involved; and (ii) the discount sharing arrangements. The expressions are presented in the text or in Appendix B. We then calculated the four relevant $\theta_{ij}^k$ with $i = A, B$, $j = A, B$ and $k = X, Y$ in each scenario, again as a function of the discounts and of the discount sharing arrangements. This allowed us to establish the upper bound for the discounts in each scenario below which all $\theta_{ij}^k$ were on $[0, 1]$ for all admissible ways of sharing the discount between firms, i.e., for all $\alpha$ on $[0, 1]$. In most cases, this threshold was obtained numerically. Finally, we analyzed the aggregate profit function of the firms offering discounts.

For the cases of discounts by firms with the same quality level, we verified that the symmetric solution of equally sharing the discount verified the first-order-condition for any level of discounts and, then, using this profit sharing arrangement, we solved numerically for the level of discounts that maximize the partner firms’ aggregate profits. When doing so for the case of bilateral bundling by pairs of firms offering similar quality products, we obtained a corner solution.

For the case of unilateral discount by firms with different quality levels, we were unable to obtain an interior solution that verified the first-order conditions and analyzed every possible case where $\beta$ and/or $\alpha$ were at a corner solution. In the case of bilateral bundling, we were able to establish (numerically) that the derivative with respect to the percentage of the discount financed by the high quality producer is always positive at a symmetric equilibrium, for any level of the discounts. Hence, with the high quality producer financing the entirety of the discount, we obtained numerically the symmetric value for the discount that verifies the first-order-conditions. In all cases, second-order conditions were verified.

Proof of Proposition 1: See text.
Appendix B - Equilibrium Prices

The expressions for the equilibrium prices when firms with the same quality level offer bundled discounts are given by

\[
\begin{align*}
\frac{P_X(\alpha_A, \beta_A, \alpha_B, \beta_B)}{s} &= \frac{P_Y(1 - \alpha_A, \beta_A, 1 - \alpha_B, \beta_B)}{s} \\
&= \frac{\alpha_A^2 \beta_A^2 ((8\beta_A + 2\beta_B + 8 + (\beta_A + \beta_B)^2) + 2\beta_B \alpha_B (\beta_A + \beta_B + 1)) + \alpha_A \beta_A (32\beta_A + 8\beta_B + 32 + 2(5\beta_B - 6\beta_A) (\beta_A + \beta_B) - 3(\beta_A + \beta_B)^3) + \alpha_A \beta_A (2\beta_B^2 \alpha_B (\beta_A + \beta_B + 1) - 4\beta_B \alpha_B (2\beta_A + 3\beta_B + (\beta_A + \beta_B)^2)) + \alpha_B^2 \beta_B^2 (2\beta_A + 4\beta_B + 4 + (\beta_A + \beta_B)^2) + \alpha_B \beta_B (4\beta_A + 16\beta_B + 8 - 2(6\beta_B - \beta_A) (\beta_A + \beta_B) - 3(\beta_A + \beta_B)^3) + 16\beta_A - 8\beta_B - 12(\beta_A + 2\beta_B) (\beta_A + \beta_B) + 8(\beta_B - \beta_A) (\beta_A + \beta_B)^2 + 96}{4(36 - (\alpha_A \beta_A + \alpha_B \beta_B)(\beta_A (1 - \alpha_A) + \beta_B (1 - \alpha_B)) - 6(\beta_A + \beta_B)^2)}
\end{align*}
\]

The expressions for the equilibrium prices when firms with different quality level offer bundled discounts are given by

\[
\begin{align*}
\frac{p_X(\alpha_{AB}, \beta_{AB}, \alpha_{BA}, \beta_{BA})}{s} &= \frac{p_Y(1 - \alpha_A, \beta_A, 1 - \alpha_B, \beta_B)}{s} \\
&= \frac{\alpha_{AB}^2 \beta_{AB}^2 ((4\beta_A + 2\beta_B + 4 + (\beta_A + \beta_B)^2) + 2\beta_B \alpha_B (\beta_A + \beta_B + 1)) + \alpha_{AB} \beta_{AB} (12\beta_A + 16 + 2(\beta_B - 6\beta_A) (\beta_A + \beta_B) - 3(\beta_A + \beta_B)^3) + \alpha_{AB} \beta_{AB} (2\beta_B^2 \alpha_B (\beta_A + \beta_B + 1) - 4\beta_B \alpha_B (3\beta_A + 2\beta_B + 1 + (\beta_A + \beta_B)^2)) + \alpha_B^2 \beta_B^2 (2\beta_A + 8\beta_B + 4 + (\beta_A + \beta_B)^2) + \alpha_B \beta_B (16\beta_A + 40\beta_B + 16 - 2(6\beta_B - 5\beta_A) (\beta_A + \beta_B) - 3(\beta_A + \beta_B)^3) + 8\beta_B - 16\beta_A - 12(\beta_A + \beta_B) - 8(\beta_B - \beta_A) (\beta_A + \beta_B)^2 + 48}{4(36 - (\alpha_A \beta_A + \alpha_B \beta_B)(\beta_A (1 - \alpha_A) + \beta_B (1 - \alpha_B)) - 6(\beta_A + \beta_B)^2)}
\end{align*}
\]

The expressions for the equilibrium prices when firms with the same quality level offer bundled discounts are given by

\[
\begin{align*}
\frac{P_X(\alpha_{AB}, \beta_{AB}, \alpha_{BA}, \beta_{BA})}{s} &= \frac{P_Y(\alpha_{BA}, \beta_{BA}, \beta_{AB}, \alpha_{AB})}{s} \\
&= \frac{\alpha_{AB}^2 \beta_{AB}^2 (8\beta_{AB} + 4\beta_{BA} + 8 + (\beta_{AB} + 3\beta_{BA}) (\beta_{AB} + \beta_{BA}) - 2\alpha_{BA} \beta_{BA} (\beta_{AB} + \beta_{BA} + 1)) + \alpha_{AB} \beta_{AB} (32\beta_{AB} + 8\beta_{BA} + 16 - 10\beta_{AB} \beta_{BA} - 12\beta_{AB}^2 - 3\beta_{AB}^3 - 5\beta_{BA}^3 - 15\beta_{AB} \beta_{BA}^2 - 15\beta_{AB}^2 \beta_{BA}) + \alpha_{AB} \beta_{AB} (4\beta_{BA} \alpha_{BA} (\beta_{AB} + 2) (\beta_{AB} + \beta_{BA}) + 2\beta_{BA} \alpha_{BA}^2 (\beta_{AB} + \beta_{BA} + 1)) + \alpha_{BA}^2 \beta_{BA}^2 (2\beta_{AB} + 4\beta_{BA} + 4 + (\beta_{AB} + \beta_{BA})^2) + \alpha_{BA} \beta_{BA} (6\beta_{AB} \beta_{BA} - 2\beta_{AB}^2 + 3\beta_{AB}^3 + 4\beta_{BA}^2 + 3\beta_{BA}^2 + 5\beta_{AB} \beta_{BA}^2 + 7\beta_{AB}^2 \beta_{BA} - 16 - 24\beta_{BA} - 4\beta_{AB}) + 8\beta_{AB} + 96 - 32\beta_{AB} \beta_{BA} - 12\beta_{AB}^2 - 8\beta_{AB}^3 - 4\beta_{BA}^4 - 6\beta_{AB}^2 \beta_{BA} - 7\beta_{AB} \beta_{BA}^3 - 3\beta_{AB}^3 \beta_{BA} - 8\beta_{AB}^2 \beta_{BA}^2}{4(-\beta_{AB} (1 - \alpha_{AB}) + \alpha_{BA} \beta_{BA})(\beta_{BA} (1 - \alpha_{BA}) + \alpha_{AB} \beta_{AB}) + 36 - 6(\beta_{AB} + \beta_{BA})^2)}
\end{align*}
\]

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\[
\begin{align*}
p_X(\alpha_{AB}, \beta_{AB}, \alpha_{BA}, \beta_{BA}) &= p_Y(\alpha_{BA}, \beta_{BA}, \beta_{AB}, \alpha_{AB}) = \\
\left(\begin{array}{c}
\alpha_{AB}^2 \beta_{AB}^2 (4 (\beta_{AB} + \beta_{BA} + 1) + (\beta_{AB} + 3 \beta_{BA}) (\beta_{AB} + \beta_{BA}) - 2 \beta_{BA} \alpha_{BA} (\beta_{AB} + \beta_{BA} + 1)) + \\
\alpha_{AB} \beta_{AB} (12 \beta_{AB} - 4 \beta_{BA} + 8 - 22 \beta_{AB} \beta_{BA} - 12 \beta_{AB} - 3 \beta_{AB}^3 - 4 \beta_{BA}^2 - 15 \beta_{AB} \beta_{BA}^2 - 13 \beta_{AB}^2 \beta_{BA}) + \\
\alpha_{AB} \beta_{AB} (4 \beta_{BA} \alpha_{BA} (3 \beta_{A} + \beta_{BA} + \beta_{AB} + \beta_{BA} + 1) + 2 \beta_{BA} \alpha_{BA} (\beta_{AB} + \beta_{BA} + 1) - 5 \beta_{BA}^3) + \\
\alpha_{BA} \beta_{BA} (3 \beta_{AB}^3 - 10 \beta_{AB} - 2 \beta_{AB} \beta_{BA}^2 - 4 \beta_{BA}^2 + 5 \beta_{AB} \beta_{BA}^2 + 7 \beta_{AB} \beta_{BA} - 48 \beta_{BA} - 16 \beta_{AB}) + \\
\alpha_{BA} \beta_{BA} (3 \beta_{AB} - 8 \beta_{AB} + 48 + 4 \beta_{AB} \beta_{BA} - 12 \beta_{AB}^2 + 8 \beta_{BA}^3 + 44 \beta_{BA}^2 - 12 \beta_{BA}^3 + \\
-2 \beta_{BA}^4 - 8 \beta_{AB} \beta_{BA}^2 + 18 \beta_{BA} \beta_{BA}^2 - 7 \beta_{AB} \beta_{BA}^2 - 3 \beta_{AB} \beta_{BA} - 8 \beta_{BA}^2 \beta_{BA} \\
\end{array}\right) \\
\left(\begin{array}{c}
4 (- (\beta_{AB} - 1 - \alpha_{AB}) + \alpha_{BA} \beta_{BA}) (\beta_{BA} (1 - \alpha_{BA}) + \alpha_{AB} \beta_{BA}) + 36 - 6 (\beta_{AB} + \beta_{BA})^2)) \\
\end{array}\right)
\end{align*}
\]

Appendix C - Welfare Analysis

Whenever discounts are offered by firms with the same quality level, consumers surplus is given by:

\[
CS = \int_0^{\theta_{AB}} \left( \int_0^{\theta_{BA}} \left( \theta_{XS_B} + \theta_{YS_B} - p_X - p_Y + (s_A - s_B) \beta_B \right) d\theta_X \right) d\theta_Y + \\
\int_0^{\theta_{BA}} \left( \int_0^{\theta_{AB}} \left( \theta_{XS_A} + \theta_{YS_B} - P_X - p_Y \right) d\theta_X \right) d\theta_Y + \\
\int_0^{\theta_{AB}} \left( \int_0^{\theta_{BA}} \left( \theta_{XS_B} + \theta_{YS_B} - p_X - p_Y + (s_A - s_B) \beta_B \right) d\theta_X \right) d\theta_Y + \\
\int_0^{\theta_{BA}} \left( \int_0^{\theta_{AB}} \left( \theta_{XS_A} + \theta_{YS_B} - P_X - P_Y + (s_A - s_B) \beta_B \right) d\theta_X \right) d\theta_Y + \\
\int_0^{\theta_{BA}} \left( \int_0^{\theta_{AB}} \left( \theta_{XS_B} + \theta_{YS_B} - P_X - P_Y \right) d\theta_X \right) d\theta_Y + \\
\int_0^{\theta_{AB}} \left( \int_0^{\theta_{BA}} \left( \theta_{XS_A} + \theta_{YS_B} - P_X - (s_A - s_B) \beta_B \right) d\theta_X \right) d\theta_Y + \\
\int_0^{\theta_{BA}} \left( \int_0^{\theta_{AB}} \left( \theta_{XS_B} + \theta_{YS_B} - P_X \right) d\theta_X \right) d\theta_Y + \int_0^{\theta_{AB}} \left( \int_0^{\theta_{BA}} \left( \theta_{XS_B} - p_Y \right) d\theta_X \right) d\theta_Y + \\
\int_0^{\theta_{BA}} \left( \int_0^{\theta_{AB}} \left( \theta_{YS_B} - p_X \right) d\theta_X \right) d\theta_Y + \int_0^{\theta_{AB}} \left( \int_0^{\theta_{BA}} \left( \theta_{YS_B} - p_Y \right) d\theta_X \right) d\theta_Y
\]

Whenever discounts are offered by firms with different quality levels, consumers surplus is given by:
\[ CS = \int_0^{\theta_{X}^{AB}} \left( \int_0^{\theta_{Y}^{BA}} \left( \theta_X s_B + \theta_Y s_B - p_X - p_Y \right) d\theta_Y \right) d\theta_X + \]

\[ \int_{\theta_{X}^{AB}}^{1} \left( \int_{\theta_{Y}^{BA}}^{\theta_{X}^{AB}} \left( \theta_X s_A + \theta_Y s_B - P_X - p_Y + (s_A - s_B) \beta_{AB} \right) d\theta_Y \right) d\theta_X + \]

\[ \int_0^{\theta_{Y}^{BA}} \left( \int_{\theta_{Y}^{BA}}^{\theta_{X}^{AB}} \left( \theta_X s_B + \theta_Y s_A - p_X - P_Y + (s_A - s_B) \beta_{BA} \right) d\theta_Y \right) d\theta_X + \]

\[ \int_{\theta_{X}^{AB}}^{1} \left( \int_{\theta_{Y}^{BA}}^{\theta_{X}^{AB}} \left( \theta_X s_B + \theta_Y s_A - p_X - P_Y + (s_A - s_B) \beta_{BA} \right) d\theta_Y \right) d\theta_X + \]

\[ \int_{\theta_{X}^{AB}}^{1} \left( \int_{\theta_{Y}^{BA}}^{\theta_{X}^{AB}} \left( \theta_X s_A + \theta_Y s_A - P_X - P_Y \right) d\theta_Y \right) d\theta_X + \]

\[ \int_{\theta_{Y}^{BA}}^{1} \left( \int_{\theta_{X}^{AB}}^{\theta_{Y}^{BA}} \left( \theta_X s_A + \theta_Y s_A - P_X - P_Y \right) d\theta_Y \right) d\theta_X + \]

\[ \int_{\theta_{Y}^{BA}}^{1} \left( \int_{\theta_{X}^{AB}}^{\theta_{Y}^{BA}} \left( \theta_X s_B - P_X \right) d\theta_X + \int_{0}^{\theta_{X}^{AB}} \left( \theta_X s_B - P_X \right) d\theta_X + \]

\[ \int_{\theta_{Y}^{BA}}^{1} \left( \int_{\theta_{X}^{AB}}^{\theta_{Y}^{BA}} \left( \theta_Y s_B - P_Y \right) d\theta_Y + \int_{0}^{\theta_{Y}^{BA}} \left( \theta_Y s_B - P_Y \right) d\theta_Y \right) \]

References


