Estimating the Tail Shape Parameter from Option Prices

Kam Hamidieh
khamidieh@fullerton.edu
Department of Mathematics
CSUF
McCarthy Hall 154
Fullerton CA 92834
January, 2012

Abstract

In this paper, a method to estimate the tail shape parameter of the risk neutral density from option prices is developed. Closed form pricing formulas for out-of-the-money European style options are derived. The pricing formulas satisfy many well known model-free no-arbitrage properties for the options. The focus is only on the tails of the risk neutral density and not on the entire body of the density as many works have already done this. The method is quite general, and applies to a large class of risk neutral densities. Our method can be used without interpolating the implied volatility, or even the knowledge of the current index value or the dividend yield or the risk free rate. This is in contrast to every other method that attempts to estimate the risk neutral density. A case study using S&P 500 Index options is given. In particular, the estimation of the tail shape of S&P 500 index just prior to the market turmoil of the September 2008 shows a thickening of the left tail but a thinning in the midst of the turmoil.

1 Introduction

Option prices contain information about the future prices of the underlying asset. A popular way to obtain this information is by estimating (or recovering) the risk neutral density directly from
option prices. Furthermore, tracking the risk neutral density can reveal how this information is evolving in time.

In their seminal work, Breeden and Litzenberger (1978) show that

\[
\text{risk neutral density} = \text{future value of a risk free bond} \times \frac{\partial^2 (\text{option})}{\partial (\text{strike})^2}.
\]


In this paper, we only focus on the tail regions and not on the body of the risk neutral density. We propose a method to recover the tail shape parameter of the risk neutral density.

The tail shape is the key parameter that quantifies the rate at which the tail of a distribution decays. A larger tail shape parameter implies larger probabilities of extreme values. The left panel of figure (1) shows two overlaid densities: a standard lognormal density, and an F density with the first and second degrees of freedom set to 5. Both densities have approximate means of 1.7. Note that the two densities appear almost indistinguishable. However, their tail probabilities are drastically different. The right panel of figure (1) shows the ratio of the tail probabilities of the F over the lognormal. The ratio diverges for larger values; for example, values greater than 14 are about 40% more likely for this F distribution as compared to the lognormal. This is because this F distribution has a larger tail shape parameter (0.4) versus the lognormal (zero.) (Appendix (A) contains a detailed discussion of the tail shape parameters related to the lognormal and F distributions.)

Tracking the tail shape parameter over time reveals how the risk neutral density’s tails are evolving in time. We highlight this application through a case study of the S&P 500 Index options during the financial crisis of 2008. Additionally, we achieve the following: we derive closed form pricing formulas for out-of-the-money European style options. We show that the pricing formulas satisfy many of the well known model-free no-arbitrage bounds for the options. We find a functional
relationship among out-of-the-money options. Lastly, we show that only out-of-the-money options are needed to estimate the tail shape; no information about the current value of the underlying or the interest rates or the dividend yields are needed.

Our method to recover the tail shape parameter is based on a well known asymptotic theorem from extreme value theory. The theorem states that the excess value of a random variable over a large threshold can be approximated by the generalized Pareto random variable. This result applies under very mild conditions to a large class of distributions; virtually all the well known continuous distributions - among them are F, t, normal, inverse normal, chi-squared, gamma, exponential, lognormal, uniform, and their additive mixtures, such as mixtures of lognormals - would satisfy the conditions of this theorem.

The value of a call option is based on the expected value of the difference of the underlying minus the strike. When the option strike is large, the difference can be approximated by a generalized Pareto random variable. This yields a pricing formula based on the first moment of a generalized Pareto random variable. Similar formulas are also obtained for the put options. The moment term contains the tail shape parameter which allows us to estimate it.

Markose and Alentorn (2011) also estimate the tail shape parameter but use the generalized extreme value distribution for the entire density of the asset returns. (We hope that the generalized extreme value distribution is not confused with the generalized Pareto distribution.) Their method
is only applicable if the entire density has the generalized extreme value distribution. In contrast, our method is applicable regardless of the underlying distribution.

The remainder of the paper is organized as follows. Section (2) reviews the key theoretical results important to this paper. In Section (3), the pricing formulas are derived. Section (4) contains the theoretical supporting material. Section (5) describes the methodology in recovering the tail shape and discusses the relevant issues. Section (6) demonstrate the accuracy of the pricing formulas for out-of-the-money options via a simulation study. Section (7) applies the method in a case study of the S&P 500 Index options. Section (8) concludes.

2 Relevant Background Theory

In this section, we briefly give an overview of the relevant theory along with a number of references.

The generalized Pareto (GP) will be the most important distribution for us:

\[
P(Y \leq y) = H_{\beta, \xi}(y) = \begin{cases} 
1 - \left(1 + \frac{y}{\beta}\right)^{-1/\xi} & \text{if } \xi \neq 0 \\
1 - \exp\left(-\frac{y}{\beta}\right) & \text{if } \xi = 0
\end{cases}
\]

where \(\beta > 0\), \(y \geq 0\) when \(\xi \geq 0\), and \(0 \leq y \leq -\beta/\xi\) when \(\xi < 0\). Here, \(\beta\) and \(\xi\) are the scale and the tail shape parameters respectively.

A random variable \(X\) is said to belong to the maximum domain of attraction (MDA) of an extreme value distribution \(G\) if for independent and identically distributed (iid) \(X, X_1, \ldots, X_n\), there exist constants \(c_n > 0\), and \(d_n \in \mathbb{R}\) such that

\[
P\left\{c_n^{-1}\left(\max(X_1, \ldots, X_n) - d_n\right) \leq x\right\} \longrightarrow G(x) = \exp\{-H_{1,\xi}(x)\}, \quad \text{as } n \to \infty.
\]

Just about any continuous distribution used in the statistics and finance literature belongs to the MDA of \(G\). See Chapter 1 of Resnick (1987) for the detailed theoretical discussions and proofs of the MDA limit theorems.

If \(X\) belongs to the MDA of \(G\), then the excess \(X - u|X > u\), when \(u\) is near the endpoint of \(X\), can be approximated by the GP distribution. This applies regardless of the original underlying distribution of the random variable. The Pickands-Balkema-De Haan theorem, taken from page
Pickands-Balkema-De Haan Theorem Suppose that $X$ with distribution function $F$ belongs to the MDA of $G$. Let $x^* = \sup\{x \in \mathbb{R} : F(x) < 1\}$ denote the upper endpoint of the distribution of $X$. Then, we have:

$$\lim_{u \to x^*} \sup_{0 \leq x \leq x^*-u} |\mathbb{P}(X - u \leq x | X > u) - H_{\beta(u),\xi}(x)| = 0,$$

for some function $\beta(u) > 0$ and a fixed shape parameter $\xi$.

The original proofs are found in Balkema and de Haan (1974) and Pickands (1975). In depth discussions of GP distribution are given in the pages 160–167, and 352–370 of Embrechts et al. (1997).

The mean of a GP random variable is:

$$\mathbb{E}[Y] = \frac{\beta}{1 - \xi}, \quad \xi < 1.$$  

An important application of the Pickands-Balkema-De Haan is in the construction of extreme quantile estimators. Let $x_q = \inf\{x \in \mathbb{R} : F(x) \geq q, q \in (0, 1)\}$ be the $q$-quantile of $X$ with the distribution function $F$. When $x_q > u$, we have:

$$1 - q = \mathbb{P}(X > x_q) = \mathbb{P}(X - u > x_q - u)$$

$$= \mathbb{P}(X - u > x_q - u | X > u) \mathbb{P}(X > u)$$

GP if Pickands-Balkema-De Haan holds.

$$\approx (1 - \xi/\beta(u)(x_q - u))^{-1/\xi}\mathbb{P}(X > u)$$

Inverting the above gives:

$$x_q \approx u + \frac{\beta(u)}{\xi} \left[ \left( \frac{\mathbb{P}(X > u)}{\mathbb{P}(X > x_q)} \right)^{\xi} - 1 \right].$$  

(3)

The approximation improves as $u \to x^*$. The estimator in equation (3) has been one of the most successful workhorses of the extreme value theory. For example, Chaves-Demoulin et al. (2005) and McNeil and Frey (2000) use equation (3) for estimating the Value-at-Risk (VaR). Equation (3) can
be re-expressed as:

\[
P(X > x_q) \approx P(X > u) \left[ \frac{\xi}{\beta(u)} (x_q - u) + 1 \right]^{-1/\xi}.
\]  
(4)

The above form will be more useful to our analysis later on.

A crucial lemma follows:

**Lemma - Closure of GP with Respect to Changes in Threshold**

Let \( X \) (or \( F \)) with right endpoint \( x^* \) belong to the MDA of \( G \). Assume that for some high threshold \( u \), and \( 0 \leq x \leq x^* - u \) we have

\[
P(X - u \leq x | X > u_0) = H_{\beta(u_0), \xi}(x),
\]

where \(-\infty < \xi < +\infty\), and \( \beta(u) > 0 \). Then for any higher threshold \( u_1 > u_0 \),

\[
P(X - u_1 \leq x | X > u_1) = H_{\beta(u_1), \xi}(x),
\]

where now

\[
\beta(u_1) = \beta(u_0) + \xi(u_1 - u_0).
\]  
(5)

See page 279 of McNeil *et al.* (2005) for the proof. Once a threshold \( u_0 \) is chosen so that \( X - u_0 | X > u_0 \) has GP distribution, then for a higher threshold \( u_1 > u_0 \), \( X - u_1 | X > u_1 \) will also have GP distribution.

### 3 Out-of-the-Money Option Pricing Formulas

Let \( S_T, K \), and \( r \) be the price of an underlying asset at expiration time \( T \), strike, and risk free rate respectively. The price of a call option \( C \) today is:

\[
C = e^{-rT} \mathbb{E}[\max(S_T - K, 0)] = e^{-rT} \mathbb{E}[S_T - K | S_T > K] \mathbb{P}(S_T > K),
\]  
(6)

where the expectation \( \mathbb{E} \) is taken under the risk neutral measure \( \mathbb{P} \). We assume the risk neutral density exists.
Let $S_T$ belong to the MDA of $G$. Assume $K$ is the strike at or above which the Pickands-Balkema-De Haan Theorem holds. Then the excess $S_T - K|S_T > K$ has an approximate GP distribution with the parameters $\beta(K)$ and $\xi$. The large strike $K$ plays the role of the high threshold $u$ in the Pickands-Balkema-De Haan Theorem.

We will assume that $\xi < 1$, which means that the first moment of $S_T - K|S_T > K$ exists. This assumption is needed to exclude the possibility of having a non-sense infinite option value. Using equations (2) and (6) we obtain:

$$C = e^{-rT} \mathbb{E}[S_T - K|S_T > K] \mathbb{P}(S_T > K) \approx e^{-rT} \frac{\beta(K)}{1 - \xi} \mathbb{P}(S_T > K) = C^*.$$  \hspace{2cm} (7)

Equation (7) gives us the first pricing formula for out-of-the-money (OTM) calls. The re-labeling by $*$ is intended to emphasize that the option price is obtained by assuming GP for $S_T - K|S_T > K$. Note that although the scale parameter $\beta(K)$ may be a function of $K$, we will not need to know the form of this function; we will treat $\beta(K)$ as a parameter. The approximation error in equation (7) vanishes as the strike increases; see Proposition (3) in section (4).

The closure property of GP, as stated in section (2), tells us that if $S_T - K_0|S_T > K_0$ has a GP distribution then $S_T - K_i|S_T > K_i$, with $K_i > K_0$, will also have GP with the same shape parameter $\xi$, and a new scale parameter based on equation (5):

$$\beta(K_i) = \beta(K_0) + \xi(K_i - K_0).$$  \hspace{2cm} (8)

Let $C_0$ and $C_i$ represent the values of the options at strikes $K_0$ and $K_i$ respectively with $K_i > K_0$. Then from equation (7) we have:

$$C_0^* = e^{-rT} \frac{\beta(K_0)}{1 - \xi} \mathbb{P}(S_T > K_0),$$  \hspace{2cm} (9)

and

$$C_i^* = e^{-rT} \frac{\beta(K_i)}{1 - \xi} \mathbb{P}(S_T > K_i).$$  \hspace{2cm} (10)

A relationship can be derived between $C_0^*$ and $C_i^*$ which eliminates the following terms: $\mathbb{P}(S_T > K_0), \mathbb{P}(S_T > K_i)$, and $e^{-rT}$. Note that each strike corresponds to a quantile of $S_T$. Referring to
equation (4), letting \( X = S_T, x_q = K_i, u = K_0 \), we have:

\[
\mathbb{P}(S_T > K_i) \approx \mathbb{P}(S_T > K_0) \left[ \frac{\xi}{\beta(K_0)} (K_i - K_0) + 1 \right]^{-1/\xi}. \tag{11}
\]

Beginning with equation (10) and substituting for the terms \( \mathbb{P}(S_T > K_i) \) and \( \beta(K_i) \) with equations (11) and (8) respectively, and rearranging we obtain:

\[
C_i^* = e^{-rT} \mathbb{E}[K'_0 - S_T | S_T < K'_\text{\footnotesize{\text{\prime}}}} \mathbb{P}(S_T < K'_\text{\footnotesize{\text{\prime}}}) \]

Equation (12) shows that for high strike values, there is a functional relationship among the values of the OTM call options. The functional relationship is parameterized by the GP parameters only. Other parameters such as the current value of the underlying or the interest rate or the time to expiration do not show up but are imbedded in the relationship.

In the case of the put options, we will be dealing with the left tail of \( S_T \). This time we assume \(-S_T\) belong to the MDA of \( G \). The price of a put \( P \) is:

\[
P = e^{-rT} \mathbb{E}[\max(K'_\text{\footnotesize{\text{\prime}}}, -S_T, 0)] = \exp(-rT) \mathbb{E}[K'_\text{\footnotesize{\text{\prime}}}, -S_T | S_T < K'_\text{\footnotesize{\text{\prime}}}] \mathbb{P}(S_T < K'_\text{\footnotesize{\text{\prime}}}). \tag{13}
\]

In this case, \( K'_\text{\footnotesize{\text{\prime}}} \) will be a low strike value in the left tail. The notation ‘ is meant to differentiate this strike from the strike \( K \) for the call options. Setting \( L = -K'_\text{\footnotesize{\text{\prime}}} \) and \( Y_T = -S_T \), we have:

\[
P = e^{-rT} \mathbb{E}[K'_\text{\footnotesize{\text{\prime}}}, -S_T | S_T < K'_\text{\footnotesize{\text{\prime}}}] \mathbb{P}(S_T < K'_\text{\footnotesize{\text{\prime}}}) \]

\[
P = e^{-rT} \mathbb{E}[Y_T - L | Y_T > L] \mathbb{P}(Y_T > L)
\]

The last line of the above equations is in the same form as equation (6). By the same line of reasoning that led to the pricing formula for the call options, the price of an OTM put option with a strike \( K'_\text{\footnotesize{\text{\prime}}} \) is:

\[
P^* = e^{-rT} \mathbb{P}(S'_T < K'_\text{\footnotesize{\text{\prime}}}). \tag{15}
\]
Similar to equation (12), for two puts \( P_i \) and \( P_0 \) with strikes \( K_0' > K_i' \), we get:

\[
P_i^* = P_0^* \left[ \frac{\xi'}{\beta(K_0')} \left( K_0' - K_i' \right) + 1 \right]^{1-1/\xi'},
\]

(16)

where \( P_0^* \) is in the same form as equation (15).

Appendix (C) contains a “toy” example which demonstrates the GP based pricing of an option.

4 Theoretical Considerations

The following two propositions show that the pricing formulas derived in section (3) do satisfy some well known bounds and properties. The first proposition concerns the calls. The proof is in Appendix (B).

Proposition 1 Let \( S_T > 0 \) with \( \mathbb{E}[S_T] < \infty \) represent the price of an asset at time \( T > 0 \). Assume that:

(a) The risk neutral density exists.

(b) \( S_T \) belongs to the MDA of \( G \).

(c) There is a strike \( K_0 \) such that \( S_T - K_0 \mid S_T > K_0 \) has GP distribution with the shape parameter \( \xi \) and the scale parameter \( \beta(K_0) > 0 \).

Then the price of a call option with a strike \( K_0 \) is:

\[
C_0^* = e^{-rT} \frac{\beta(K_0)}{1 - \xi} \mathbb{P}(S_T > K_0),
\]

(17)

and the price of a call option with a strike \( K > K_0 \) is:

\[
C^* = C_0^* \left[ \frac{\xi}{\beta(K_0)} \left( K - K_0 \right) + 1 \right]^{1-1/\xi}.
\]

(18)

Furthermore, the following properties hold:

1. \( C^* \geq 0 \).

2. When \( \xi = 0 \), then \( C^* = C_0^* e^{-(K-K_0)/\beta(K_0)} \).

3. As \( K \uparrow \infty \), holding all other parameters fixed, \( C^* \downarrow 0 \).
4. $C^*$ is a convex function of $K$.

5. As $\xi \uparrow 1$, holding all other parameters fixed, $C^* \uparrow \infty$.

Assumption (a) ensures that the pricing formula in equation (6) holds. Assumptions (b), and (c) allow us to express the (conditional) expected value in equation (6) in terms of an expected value of a GP random variable.

The price of a call option can never be negative; part (1) shows that our pricing formula can never be negative. Part (2) shows that when the tail of the risk neutral density (RND) drops exponentially fast - for example, this would be the case when $S_T$ has lognormal density - then the price of an OTM call drops exponentially fast as well. It is a well known fact that call options become less valuable with higher strikes. Part (3) shows that our pricing formula retains this property. Any call option is a convex function of the strike; part (4) shows that this is valid for our pricing formula. Part (5) shows that as the right tail becomes heavier, an OTM call increases in value; the tail shape parameter plays a similar role to the volatility.

The next proposition concerns the puts. It is stated separately from the proposition for the calls because the left and the right tails may behave differently. The proof of this proposition is similar to the proof of Proposition (1) and thus omitted.

**Proposition 2** Let $S_T > 0$ with $\mathbb{E}[S_T] < \infty$ represent the price of an asset at time $T > 0$. Assume that:

(a) The risk neutral density exists.
(b) $-S_T$ belongs to the MDA of $G$.
(c) There is a strike $K'_0$ such that $K'_0 - S_T \mid S_T < K'_0$ has GP distribution with the shape parameter $\xi'$ and the scale parameter $\beta(K'_0) > 0$.

Then the price of a put option with a strike $K'_0$ is:

$$P_0^* = e^{-rT} \frac{\beta(K'_0)}{1-\xi'} \mathbb{P}(S_T < K'_0),$$

and the price of a put option with a strike $K' < K'_0$ is:

$$P^* = P_0^* \left[ \frac{\xi'}{\beta(K'_0)} \left( K'_0 - K' \right) + 1 \right]^{1-1/\xi'}.$$

10
Furthermore, the following properties hold:

1. \( P^* \geq 0 \).

2. As \( \xi' \uparrow 0 \), then \( P^* = P^*_0 e^{-(K'_0 - K')/\beta(K'_0))} \).

3. As \( K' \downarrow 0 \), holding all other parameters fixed, \( P^* \downarrow 0 \).

4. \( P^* \) is a convex function of \( K' \).

5. As \( \xi' \downarrow -\infty \), holding all other parameters fixed, \( P^* \downarrow 0 \).

The interpretation of the Proposition (2) is analogous to the Proposition (1) except we need not consider the case \( \xi \to 1 \), as the left endpoint of \( ST \) is finite.

The following proposition shows that the relative error in our approximation tends to zero with higher strikes for the calls and lower strikes for the puts. The proof is in the Appendix (B).

**Proposition 3** Let \( ST > 0 \) with \( \mathbb{E}[ST] < \infty \) represent the price of an asset at time \( T > 0 \). Let both \( ST \) and \( -ST \) belong to the MDA of \( G \). Assume the risk neutral density exists. Let \( C^*, P^*, C, \) and \( P \) be defined as in equations (18), (20), (6), and (13) respectively. Then

\[
\lim_{K \to \infty} \frac{|C - C^*|}{C} = 0, \quad \text{and} \quad \lim_{K' \to 0} \frac{|P - P^*|}{P} = 0.
\]

5 Recovering the Tail Shape

Suppose there are \( n \) market (MKT) OTM calls \( C^\text{MKT}_i > C^\text{MKT}_{i+1} \) with strikes \( K_i < K_{i+1} \), \( C^\text{MKT}_0 = C^* \), \( i = \{0, \ldots, n - 1\} \), for which the GP assumption is valid. Likewise for the OTM puts, suppose we have \( P^\text{MKT}_0 = P^*_0 \) with \( K'_0 \), \( n' \) puts \( P^\text{MKT}_j < P^\text{MKT}_{j+1} \), and \( K'_j > K'_{j+1} \), \( j = \{0, \ldots, n' - 1\} \).

A practical issue is the choice of the initial strikes \( K_0 \) and \( K'_0 \) (or analogously the initial call \( C^\text{MKT}_0 = C^* \) and the put \( P^\text{MKT}_0 = P^*_0 \)). One of the most commonly used techniques is to choose a small percentage of the data in the tails. For example, McNeil and Frey (2000) and Chaves-Demoulin et al. (2005) show excellent results while selecting 10% of the data in the tails. We will also follow this simple technique. When recovering the right tail shape parameter, we will choose an initial call such that 20% to 10% of the cheapest calls are selected. Similarly, when recovering
the left tail shape parameter, we will use 20% to 10% of the cheapest puts. However, the choice of the percentage of the OTM options has to be balanced against the actual number of options left for the estimation. We recommend a minimum of 5 options. We will demonstrate this in a practical case in Section (7).

Equations (12) and (16) have put us in the non-linear regression framework. The most common method of estimating $\xi$, $\beta(K_0)$, $\xi'$, and $\beta(K'_0)$ is via the least squares. However, motivated by Proposition (2), we focus on minimizing the absolute relative errors:

$$\min_{(\xi, \beta(K_0)) \in (\mathbb{R} \times \mathbb{R}^+)} \sum_{i=0}^{n-1} \left| \frac{C_{i}^{\text{MKT}} - C_{i}^{*}}{C_{i}^{\text{MKT}}} \right|,$$

and

$$\min_{(\xi', \beta(K'_0)) \in (\mathbb{R} \times \mathbb{R}^+)} \sum_{j=0}^{n'-1} \left| \frac{P_{j}^{\text{MKT}} - P_{j}^{*}}{P_{j}^{\text{MKT}}} \right|.$$  \hfill (21)

(Our own experimentation with the least squares method yielded similar results; see Appendix (D) for an intuitive explanation.)

The model errors are defined as:

$$e_i = \frac{C_{i}^{\text{MKT}} - C_{i}^{*}}{C_{i}^{\text{MKT}}}, \quad \text{and} \quad e'_j = \frac{P_{j}^{\text{MKT}} - P_{j}^{*}}{P_{j}^{\text{MKT}}}.$$  \hfill (22)

The estimates obtained by the minimization are strongly consistent under certain regularity conditions; the most relevant condition is that $(K_i, e_i)$’s, and separately $(K'_j, e'_j)$’s are iid. For details, see Assumptions A and Theorem (1) in Khoshgoftaar et al. (1992).

We use the bootstrap method to assess the uncertainty in the parameter estimates. The following bootstrap algorithm is outlined for the calls but it applies to the puts as well:

1. Let $l = 1$.

2. Select $n - 1$ cases with replacement from $(C_{1}^{\text{MKT}}, K_1), \ldots, (C_{n-1}^{\text{MKT}}, K_{n-1})$.

3. Using the minimization of the calls in equation (21), estimate $\xi$ and $\beta(K_0)$ from the resampled data. Label these new estimates as $\hat{\xi}_l$ and $\hat{\beta}(K_0)_l$.

4. Increment $l$ by 1.

5. Repeat steps (2) to (4) $m$ times, where $m$ is a predetermined number of bootstrap samples to be taken.
The result will be \( m \) pairs of parameter estimates: \((\tilde{\xi}_1, \tilde{\beta}(K_0)_1), \ldots, (\tilde{\xi}_m, \tilde{\beta}(K_0)_m)\). For a fixed \( \alpha \) (for example \( \alpha = 0.025 \)), a 100\((1 - 2\alpha)\)% confidence interval (CI) for the parameters can be constructed by taking the 100\(\alpha\) and 100\((1 - \alpha)\) quantiles of \( \tilde{\xi}_l \), and \( \tilde{\beta}(K_0)_l \), \( l = 1, \ldots, m \). For the tail shape, we have:

\[
\begin{align*}
\text{lower } 100(1 - 2\alpha)\% \text{ CI limit} &= 100(\alpha) \text{ quantile of } \tilde{\xi} \text{ distribution}, \\
\text{upper } 100(1 - 2\alpha)\% \text{ CI limit} &= 100(1 - \alpha) \text{ quantile of } \tilde{\xi} \text{ distribution}. \\
\end{align*}
\] (23)

There is an advantage to bootstrapping the cases rather than the estimated versions of the residuals in equation (22); bootstrapping the cases is less sensitive to the assumptions that the model has the correct form, and that the errors are independent of the strikes. See pages 113-115 of Efron and Tibshirani (1993) for the detailed discussions on bootstrapping the cases.

6 A Simulation Study

We generate OTM option prices from three different risk neutral densities, and then we compare them to the GP based OTM option prices.

Under BSM, the RND of \( \log(S_T) \) at the expiration time \( T \) with a known continuous dividend yield of \( y \) is:

\[
\log(S_T) \sim \mathcal{N}(\log(S_0) + (r - y - \sigma^2/2)T, \sigma^2 T),
\] (24)

where \( S_0 \) is the current value, \( \sigma \) is the volatility, \( r \) is the risk free rate, and \( \mathcal{N} \) stands for normal.

The BSM call and put prices with the strike \( K \) (see page 149 of Musiela and Rutkowski (2005)) are:

\[
C^{BSM} = S_0 e^{-yT} \Phi(d_1) - Ke^{-rT} \Phi(d_2), \quad \text{and} \quad P^{BSM} = Ke^{-rT} \Phi(-d_2) - S_0 e^{-yT} \Phi(-d_1)
\] (25)

\[
d_1 = \frac{\ln(S_0/K) + (r - y + \sigma^2/2)T}{\sigma \sqrt{T}}, \quad d_2 = \frac{\ln(S_0/K) + (r - y - \sigma^2/2)T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T},
\]

where \( \Phi \) is the cumulative distribution function of a standard normal random variable. In the simulation study, the parameters are set as follows: \( S_0 = 1000, r = 0.03, y = 0.02, \sigma = 0.2, \) and \( T = 50/365 \).
The second distribution we considered is the generalized Beta (GB) distribution. Assume the random variable $B$ has a beta distribution with the parameters $v > 0$ and $w > 0$. Then $Z \ge 0$, as defined below, with the parameters $a > 0$, $b > 0$, $v$, and $w$ has the GB distribution:

$$Z = b\left(\frac{B}{1-B}\right)^{1/a} \sim \text{GB}(a, b, v, w).$$

GB is a flexible distribution which subsumes many other distributions; see figure (1) of Bookstaber and McDonad (1987). Liu et al. (2007) use the Generalized Beta to model the RND. McDonald and Bookstaber (1991) show that if the RND is GB, then the price of the option as follows:

$$C^{GB} = S_0(\mathbb{P}(Z_1 > K)) - Ke^{-rT}(\mathbb{P}(Z_2 > K)),$$

where $Z_1 \sim \text{GB}(a, b, v + 1/a, w - 1/a)$, and $Z_2 \sim \text{GB}(a, b, v, w)$. The Martingale condition dictates that:

$$S_0 = e^{-rT}b\alpha B(v + 1/a, w - 1/a) B(v, w),$$

where $B$ is the beta function. In the simulation study, the parameters are set as follows: $a = 10$, $b = 1000$, $w = 2.85$, $w = 2.85$, $r = 0.03$, and $T = 50/365$. Based on these values, the martingale condition gives $S_0 = 1000.086$. The put values are obtained by using the the put-call parity.

The third distribution we consider is a mixture of two lognormals. The distribution is bimodal. This is another popular way to model the RND. See for example Melick and Thomas (1997). Let $W_1$, and $W_2$ be two lognormal random variables:

$$\log(W_1) \sim \mathcal{N}(\mu_1, \sigma_1^2), \text{ and } \log(W_2) \sim \mathcal{N}(\mu_2, \sigma_2^2).$$

The mixture is then:

$$\text{mixture distribution} = \alpha f_{W_1} + (1 - \alpha) f_{W_2},$$

where $0 < \alpha < 1$, and $f_{W_1}$, $f_{W_2}$ are the densities of $W_1$ and $W_2$ respectively. The martingale condition (see pages 389–393 of Jondeau et al. (2007)) is:

$$S_0 = e^{-rT}(\alpha \exp(\mu_1 + \sigma_1^2/2) + (1 - \alpha) \exp(\mu_2 + \sigma_2^2/2)).$$
Figure 2: The three risk neutral densities in the simulation are superimposed.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Mean</th>
<th>SD</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.80</th>
<th>0.85</th>
<th>0.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single Lognormal</td>
<td>1001</td>
<td>74</td>
<td>908</td>
<td>925</td>
<td>938</td>
<td>1063</td>
<td>1078</td>
<td>1098</td>
</tr>
<tr>
<td>Generalized Beta</td>
<td>1004</td>
<td>92</td>
<td>891</td>
<td>912</td>
<td>928</td>
<td>1077</td>
<td>1097</td>
<td>1122</td>
</tr>
<tr>
<td>Mixture Lognormal</td>
<td>987</td>
<td>92</td>
<td>859</td>
<td>879</td>
<td>897</td>
<td>1069</td>
<td>1083</td>
<td>1100</td>
</tr>
</tbody>
</table>

Table 1: The columns with the headings “Mean” and “SD” give the mean and standard deviation of the RND at the expiration time. The columns with the number headings between 0 and 1 give a few of the small and large quantiles of the distribution. For example, the value of 925 under the column heading “0.15”, and crossed with the row “Single Lognormal” is the value of \( x \) such that \( P(\text{Single Lognormal} \leq x) = 0.15 \).

The value of a call options with a strike \( K \) is simply a weighted average of two BSM calls with two different underlying lognormal densities:

\[
C_{\text{MIX}} = \alpha(C_{\text{BSM}} \text{ with } W_1 \text{ density}) + (1 - \alpha)(C_{\text{BSM}} \text{ with } W_2 \text{ density}).
\]

The put values are again obtained by using the the put-call parity.

We will set the parameters to the following values: \( \alpha = 0.40, \mu_1 = 6.80, \mu_2 = 6.95, \sigma_1 = 0.065, \sigma_2 = 0.055, \) and \( r = 0.03 \). The martingale condition is computed to be \( S_0 = 982.6889 \).

The three densities are overlaid in figure (2). Additional summary information about the risk neutral densities are given in the table (1).

The simulation steps and their discussions follow:
1. Select three initial high strikes corresponding to \( q \in \{0.80, 0.85, 0.90\} \) quantiles of the RND. Let these values be \( K_0(q) \). For example, in the case of lognormal from table (1) we have: \( K_0(q = 0.80) = 1063, K_0(q = 0.85) = 1078, \) and \( K_0(q = 0.90) = 1098 \). Similarly, select three initial low strikes corresponding to \( q' \in \{0.20, 0.15, 0.10\} \) quantiles of the RND. Let these values be \( K'_0(q') \). For example, for the lognormal from table (1) we have: \( K'_0(q' = 0.20) = 938, K'_0(q' = 0.15) = 925, \) and \( K'_0(q' = 0.10) = 908 \). The reason we pick three low and three high strikes is to analyze the sensitivity of the GP based option prices to the different initial options (or initial strikes alternatively.)

2. Generate 10,000 data points, \( S_i, i = 1, \ldots, 10000 \), from the RND.

3. Find the excess values:

Sets indexed by \( q = \{S_i - K_0(q) : \forall S_i > K_0(q)\} \),

Sets indexed by \( q' = \{K'_0(q') - S_i : \forall S_i < K'_0(q')\} \).

Sets indexed by \( q \) approximate the distribution of \( S_T - K_0 | S_T > K_0 \) in equation (6). Sets indexed by \( q' \) approximate \( K'_0 - S_T | S_T < K'_0 \) in equation (14). Note on average 2000, 1500, and 1000 excess values are collected when \( q = 0.80, 0.85, \) and 0.90 respectively. The same holds for the \( q' \) values.

4. Estimate the GP parameters of the excess values in the right and the left tails via MLE. (See Section (6.5) of Embrechts et al. (1997) for the MLE equations and the associated discussions for GP. We resort to MLE estimation because the exact values of the tail shape and scale parameters for a fixed thresholds of our densities are not known.) At the end of this step, we will have 3 pairs of \( (\hat{\beta}(K_0(q)), \hat{\xi}(q)) \) GP parameter estimates for the call pricing and likewise 3 pairs \( (\hat{\beta}(K'_0(q')), \hat{\xi}'(q')) \) for the put pricing. The hat designation indicates that these values are MLE based estimates.

5. Estimate the initial GP based calls via equation (7):

\[
\hat{C}_0^*(q) = \frac{\hat{\beta}(K_0(q))}{1 - \hat{\xi}(q)} \hat{p}(S_T > K_0(q)) \exp(-(r - y)T).
\]
(Note that \( y = 0 \) for the GB and mixture distributions.) Similarly, for the puts using equation (15) obtain:

\[
\hat{P}^* = \frac{\hat{\beta}(K'_0(q'))}{1 - \hat{\xi}'(q')} \hat{\mathbb{P}}(S_T < K'_0(q')) \exp(-(r - y)T).
\]

The terms \( \hat{\mathbb{P}}(S_T > K_0(q)) \) and \( \hat{\mathbb{P}}(S_T < K'_0(q')) \) are estimated by the proportion of the data that fall above \( K_0(q) \) and below \( K'_0(q') \) respectively. At the end of this steps, three initial calls and three initial puts are estimated.

6. For each \( K_0(q) \) and \( \hat{\beta}(K_0(q)), \hat{\xi}(q) \), estimate the OTM calls via equation (12) at strikes \( K_i = K + (i-1), \ i = 1, 2, \ldots, 101 \). The value of \( K \) is set above the largest initial strikes \( K_0(q) \); we choose \( K = 1100 \) for the lognormal so the range for \( K_i \)'s is \{1100, \ldots, 1200\}, \( K = 1150 \) for GB with the range \{1150, \ldots, 1250\}, and \( K = 1125 \) for the mixture with \{1125, \ldots, 1225\}. The successive calls are:

\[
\hat{C}^*_i(q) = \hat{C}^*_0(q) \left[ \frac{\hat{\xi}(q)}{\hat{\beta}(K_0(q))} \left( K_i - K_0(q) \right) + 1 \right]^{1-1/\hat{\xi}(q)}.
\]

Similarly, via equation (16) estimate puts at strikes \( K'_j = K' + (j-1), \ j = 1, 2, \ldots, 101 \). The value of \( K' \) is set below the smallest initial strike; we pick \{900, \ldots, 800\} for the lognormal, \{850, \ldots, 750\} for the GB, and \{850, \ldots, 750\} for the mixture. The successive puts are:

\[
\hat{P}^*_i(q') = \hat{P}^*_0(q') \left[ \frac{\hat{\xi}'(q')}{\hat{\beta}(K'_0(q'))} \left( K'_0(q') - K'_j \right) + 1 \right]^{1-1/\hat{\xi}'(q')}.
\]

We will now have three GP based call estimates, which started from three different initial calls, to compare with one theoretical call value. The same holds for the puts.

7. Repeat Steps (2) to (6) 20,000 times and collect all the estimated values.

The estimated option prices are now compared with the theoretical prices via the root mean square error (rmse):

\[
\text{rmse calls} = \sqrt{\frac{1}{20000} \sum_{i=1}^{20000} (C_{\text{Theoretical}} - \hat{C}^*_i(q))^2}/20000,
\]
\[
\text{rmse puts} = \sqrt{\frac{\sum_{i=1}^{20000} (P_{\text{Theoretical}} - \hat{P}^*_i(q'))^2}{20000}},
\]

The superscript “Theoretical” can be the price based on BSM, GB or MIX.

The rmse values versus the strikes are plotted in figure (3). The top, middle, and bottom rows correspond to the lognormal, the GB, and the mixture respectively. The rmse of the calls are plotted in the left panels. The rmse values of the puts are in the right panels. See Appendix (E) for a table format of the rmse values.

We can make the following summaries based on figure (3):

- In all cases except one (puts for GB), the call rmse’s are lower for the higher initial strikes, and the put rmse’s are lower for the lower initial strikes. For example, for the lognormal puts (first row, right panel of figure (3)) the rmse for \( q = 0.9 \) is less than the rmse for \( q = 0.85 \) which is less than the rmse for \( q = 0.80 \). This is expected since the quality of the GP approximation to the excess distribution improves with the increasing strikes for the right tail and decreasing strikes for the left tail.

- With the exception of the mixture of lognormals, the pricing errors are comparable regardless of the initial values of \( K_0(q) \)’s and \( K'_0(q) \)’s.

- Overall, the results show that the GP based pricing estimates the theoretical prices well. Its performance is not as good in the mixture case unless a very high initial strike in the right tail or a very low initial strike in the left tail are chosen.

We used the software package R (see R Development Core Team (2009)) in our simulations. We also used the packages developed by Graf and Nedyalkova (2011) for simulating GB random variables, and the package by S. and Stephenson (2010) for GP MLE computations. Appendix (F) contains all the simulation code.
Figure 3: The top, middle, and bottom rows correspond to the lognormal, Generalized Beta, and the mixture of lognormals respectively. In the left panels, three rmse values of out-of-the-money calls are plotted against the high strike values. In the left panels, out-of-the-money put rmse values are plotted. Each line within each plot corresponds to a different starting initial; the initial strikes for the calls were set to the 0.80, 0.85, and 0.90 quantiles of the RND and 0.20, 0.15, and 0.10 for the puts.
7 A Case Study of the S&P 500

In this section, we estimate and interpret the left tail shape parameter of the S&P 500 Index RND prior and during the financial turmoil of 2008. This study is descriptive in nature. See also Birru and Figlewski (2010) for a high frequency analysis of the S&P 500 Index RND during the turmoil.

7.1 Data Description and Preparation

The data consists of CBOE put options on the S&P 500 Index. The options expire on December 20, 2008. The start date of our data is June 23 which followed the June 22 expiration date. We did not use puts expiring in September 20. This was done to ensure that the data spanned the tumultuous months of September and October. Options too close to the expiration date often lose their liquidity. To avoid liquidity issues, Mandler (2003) on page 103 suggests between one week and one month as the required minimum time to expiration. We picked an end date of November 21, 2008, which is approximately 1 month before the expiration date of December 20. There are a total of 108 days. The average of the bid and ask for each put is taken to obtain a single price. All puts with the bid price of zero are removed.

The data are downloaded from OptionMetric in the WRDS database. The specific WRDS site is http://wrds-web.wharton.upenn.edu/wrds/ds/optionm/index.cfm. Detailed description of the data can be found in OptionMetrics (2008). We also downloaded the dividend adjusted S&P 500 Index for June 23 to November 21 from Bloomberg.

7.2 Recovering the Tail Shape from the S&P 500 Put Options

Each day 10, 15, and 20% of the lowest strike puts are selected for recovering the tail shape parameter as described in section (5). Figure (4) shows the number of puts used in recovering the left tail shape parameter. The number of available options increased during the period of the study. The sample sizes were as low as 6 when selecting only 10% of the data and as high as 35 when selecting 20%.

Figure (5) shows that the pattern for the evolution of the tail shape parameter is the same regardless of the proportions used. Our interpretation of the tail shape (at least its evolution in time) is not dependent on the proportion of the data used. We focus on the 15% results for the
Figure 4: From left to right, the number of puts when 10, 15, and 20% of the data are selected.

<table>
<thead>
<tr>
<th>Strike</th>
<th>600</th>
<th>700</th>
<th>750</th>
<th>800</th>
<th>850</th>
<th>900</th>
<th>950</th>
<th>975</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Put</td>
<td>0.08</td>
<td>0.23</td>
<td>0.25</td>
<td>0.55</td>
<td>0.75</td>
<td>1.50</td>
<td>2.55</td>
<td>3.40</td>
<td>4.60</td>
</tr>
</tbody>
</table>

Table 2: The lowest 15% OTM put option prices and their strikes for September 8, 2008.

rest of our analysis.

We first highlight the analysis for a single day of September 8, 2008. There is no special reason for the choice of this date; it is the middle point in our data. There were 59 puts for this date. The left panel of figure (6) shows a plot of the put prices versus their strikes. The 15% lowest puts, or a total of 9 puts, i.e. $59 \times 0.15 \approx 9$, (including the initial put) to the left of the vertical dotted line are used for recovering the tail shape. The 9 puts are also listed in table (2). The right panel of figure (6) shows the same lowest 15% puts with a fitted curve using the estimated values of the tail shape and scale parameters. The fitted curve is simply obtained by plugging in the estimated values of the tail shape ($\xi' = 0.0701$), scale ($\beta' = 77.0734$), and the initial put value ($\$4.60$) into equation (16), and evaluating the equation across the strike range of 600 to 1000.

As seen in figure (7), prior to the September, the RND’s left tail was thickening. This implies that extreme down movements were becoming more likely to occur. There is no fixed date in September from which the crisis is considered to begin. The two major events were the bankruptcy filing of Lehman (September 15), and the initial voting down of the TARP (Troubled Asset Relief Program) by the House of Representatives (September 29). As seen in figure (7), at the start of the September, S&P 500 began declining. The tail shape parameter showed a decreasing trend, indicating a thinner tail going into the crisis. This seems counterintuitive. However, a possible
Figure 5: The series corresponding to the left tail shape parameter estimates of the RND of the S&P 500 are shown.

Figure 6: Left Panel: Plot of the puts versus the strike for September 8, 2008. 15% of the lower puts, to the left of the vertical dotted line, are used to recover the tail shape parameter for September 8. Right Panel: The fitted line to the 15% of the lower puts is shown.
Figure 7: The top portion shows the S&P 500 Index and the bottom portion shows the left tail shape parameter when 15% of puts are used.

explanation is that even though S&P 500 was plummeting, the tail shape implied that there was a lower bound to which S&P 500 may drop to. (Distributions with a negative tail shape have a finite end point.) During the October, S&P 500 is quite volatile. The major events happening during this time were the passing of TARP (October 2), the government equity buying of major banks (October 14), the single largest percentage loss of S&P 500 of 9% since the crash of October 19, 1987 (October 16), and the large Federal Reserve interest rate cut (October 29). The tail shape parameter reflected this volatility as well but there was no trend. In the November 2008, S&P 500 declined sharply again, and there was almost a level shift in the tail shape value. Two issues make the interpretation of the tail shape more difficult during November: (1) the values of the tail shape fluctuate rapidly, and (2) the liquidity of the OTM options could have been drying up (despite taking the cautionary advice of Mandler (2003)).

In figure (8) the tail shape parameter with the 95% confidence intervals are shown. The confidence intervals were obtained based on the bootstrap method discussed in section (5) with \( m = 1000 \). Overall, the confidence intervals are narrow - which help in interpreting the evolution of the tail shape - but even narrower in later days of the study. This is partly due to the fact that higher number of puts were available for the later period.
Figure 8: Left Panel: Tail shape parameter estimates with 95% confidence interval based on the bootstrap method.

All the R code for our analysis is in Appendix (G). Unfortunately, the data can not be shared without authorized access to WRDS. However, in Appendix (H) we give a sample analysis of the tail shape estimation when synthetic BSM options are used. Appendix (I) shows how to price a new option based on the available options on September 8, 2008.

8 Conclusion

In this paper, we have proposed a method to estimate the tail shape parameter of the risk neutral density. This is a critical parameter for assessing the behavior of the tails of the risk neutral density in two important ways: (1) Tracking it can provide potentially useful information during tumultuous times. The thickening of the tail indicates increased probabilities of extreme moves for the underlying asset in the near future. (2) The tail shape parameter is an input to our theoretically justified pricing formulas for the OTM options.

Furthermore, our method has the following advantages: (1) It can be used without interpolating the implied volatility, or even the knowledge of the current index value or the dividend yield or the risk free rate. This is in contrast to every other method that attempts to obtain information from the risk neutral density implied by options. (2) It is applicable for a large class of risk neutral
densities. On the downside, some benefits to obtaining the entire density are lost; for example, our method would not detect bi-modality if it existed.

Our future research plans include answering the following questions: (1) What is the relationship between the tail shape parameters of the risk neutral density and the real density? (2) Is there a suitable transformation function between the tail shape of the risk neutral density and the real density? (3) How can the results be extended to the American type options? (4) Can the pricing formulas be applied to options with long maturity times when only thinly traded options are available?

References


29


A Tail Shape Parameters for Lognormal and F Distributions

To see that the tail shape parameter of the lognormal distribution is zero, refer to pages 147 and 152 of Embrechts et al. (1997). To show that a random variable $X$ with an F distribution and both degrees of freedom equal to 5 has a tail shape value of $2/5$, we need to demonstrate that $X$ is tail equivalent to a Pareto distribution with tail probability $x^{-1/\xi} = x^{-5/2}$, $x > 4$, so $\xi = 2/5 = 0.40$. That is

$$\lim_{x \to \infty} \frac{P(X > x)}{x^{-5/2}} = c,$$

where $0 < c < \infty$. For further discussions on the idea of tail equivalence see page 129 of Embrechts et al. (1997) or page 67 of Resnick (1987). First note:

$$P(X > x) = \gamma \int_x^\infty \frac{u^{3/2}}{(1 + (5/2)u)5} du,$$

where the expression inside the integral is just the density of $X$ and $\gamma > 0$ is a normalizing constant. Tail equivalency as follows:

$$\lim_{x \to \infty} \frac{P(X > x)}{x^{-5/2}} = \lim_{x \to \infty} \frac{-\gamma x^{1/2}}{(1 + (5/2)x)^5} = \lim_{x \to \infty} \frac{\gamma x^5}{(5/2)(1 + (5/2)x)^5} = \text{some positive constant call it } c.$$

Pareto belongs to the MDA of generalized extreme value distribution. By Proposition 3.3.9 of Embrechts et al. (1997), page 132, $X$ also belongs to the MDA of generalized extreme value distribution with the same tail shape parameter of a Pareto random variable with $\xi = 0.4$.

B Proofs

Proof of Proposition (1)

Proof. Equations (17) and (18) in the proposition statement are equivalent to equations (7) and (12) respectively, and were derived in Section (3). Let $X = S_T$, $X_q = K$, $u = K_0$ in equation (4),
by our assumption (c), the excess is GPD:

\[ \mathbb{P}(S_T > K) = \mathbb{P}(S_T > K_0) \left( \frac{\xi}{\beta(K_0)} (K - K_0) + 1 \right)^{-1/\xi}. \]

Since

\[ \frac{\mathbb{P}(S_T > K)}{\mathbb{P}(S_T > K_0)} = \frac{\mathbb{P}(S_T > K | S_T > K_0)}{\mathbb{P}(S_T > K_0)} = \left( \frac{\xi}{\beta(K_0)} (K - K_0) + 1 \right)^{-1/\xi}, \]

where \( K > K_0 \), and the above equation holds regardless of the sign of \( \xi \). Thus, we have:

\[ 0 < \left( \frac{\xi}{\beta(K_0)} (K - K_0) + 1 \right)^{-1/\xi} < 1, \quad (26) \]

or alternatively

\[ \left( \frac{\xi}{\beta(K_0)} (K - K_0) + 1 \right) > 0. \quad (27) \]

The existence of the first moment of \( S_T \) implies that \( \xi < 1 \). Since in this case we are dealing with an infinite right end point (the support of \( S_T \) is \((0, \infty)\)), the tail shape parameter is non-negative; we have: \( 0 \leq \xi < 1 \). (Note, however, our results would apply even if \( \xi < 0 \).)

For part (1), note that all the terms in equation (17) are positive implying that \( C^*_0 \) is positive. Therefore, all the terms in equation (18) are positive where the positivity of the term \( \left[ \frac{\xi}{\beta(K_0)} (K - K_0) + 1 \right]^{1-1/\xi} \) is due to the positivity of the terms in equations (26) and (27). \( C^* = 0 \) only zero if \( S_T < K \) at the time \( T \). For part (2), let \( h = 1/\xi \), so that as \( \xi \to 0^+, h \to +\infty \). The final result would not change if the limit operations is from the left. Rewriting equation (12) with \( h \), we obtain

\[ C^* = C^*_0 \left( 1 + \frac{(K - K_0)}{\beta(K_0) h} \right)^{1-h} = C^*_0 \left( \left( 1 + \frac{(K - K_0)}{\beta(K_0) h} \right)^h \right)^{-1} \left( 1 + \frac{(K - K_0)}{\beta(K_0) h} \right). \]

Taking the limits of the above, we get:

\[ \lim_{h \to +\infty} C^* = C^*_0 e^{-(K_2 - K_1)/\beta(K_1)}. \]

For part (3), to show that \( C^* \) decreases to zero, define the function \( g \) as follows:

\[ g(K) = \left( \frac{\xi}{\beta(K_0)} (K - K_0) + 1 \right)^{1-1/\xi}. \]
Taking the derivative of $g$ with respect to $K$, we get:

$$
\frac{dg}{dK} = (\xi - 1)(1/\beta(K_0)) \left( \frac{\xi}{\beta(K_0)}(K - K_0) + 1 \right)^{-1/\xi}.
$$

The derivative is negative: the first term $\xi - 1 < 0$, $\beta_0 > 0$, and the last term is positive from equation (26). Therefore $g$ as a function of $K$ is strictly decreasing and positive. Since $C^* = C_0^*g(K)$, this shows that $C^*$ is a strictly decreasing function of $K$ with a minimum of zero. Therefore $C^* \downarrow 0$, as $K \uparrow \infty$. For part (4), beginning with equation (12), and taking two partial derivatives with respect to $K$, we get:

$$
\frac{\partial^2 C^*}{\partial K^2} = (C_0^*)(1/\beta(K_0))^2 \left( \frac{\xi}{\beta(K_0)}(K - K_0) + 1 \right)^{-1/\xi - 1} (1 - \xi).
$$

All the terms in the right hand side of the above equation are greater than zero which implies that

$$
\frac{\partial^2 C^*}{\partial K^2} > 0,
$$

and thus $C^*$ is a convex function of $K$. To prove part (5), note as $\xi \uparrow 1$, holding all else fixed, then $1/(1 - \xi) \uparrow +\infty$, and $C^* \uparrow \infty$ in equation (18). □

Proof of Proposition 3

Proof. We’ll show that $\frac{C^*}{C} \to 1$, so that $\frac{|C - C^*|}{C} \to 0$, as $K \to \infty$. Using equations (7) and (6), the ratio becomes:

$$
\frac{C^*}{C} = \frac{\beta(K)}{(1 - \xi)\mathbb{E}[S_T - K|S_T > k]} = \frac{1}{(1 - \xi)\mathbb{E}[(S_T - K)/\beta(K)|S_T > k]},
$$

(28)

Let $X_K = (S_T - K)/\beta(K)|S_T > k$. We just need to show that $\lim_{K \to \infty} \mathbb{E}[X_K] = 1/(1 - \xi)$. By our assumption as $K \to \infty$, we have:

$$
X_K \overset{D}{\to} Y,
$$

where the designation $D$ is for convergence in distribution, and $Y$ is a GP random variable with parameters $\xi$ and $\beta = 1$. The form of this convergence is expressed explicitly, for example, on page 160 of Embrechts et al. (1997). Note that $X_K \leq S_T/\beta(K)$. Furthermore, $X_K$ and $S_T/\beta(K)$ are integrable random variables by our assumptions that $\mathbb{E}[S_T] < \infty$. Therefore, $X_K$ is a uni-
formly integrable sequence. See the bottom of page 338 of Billingsley (1995). The convergence in distribution of uniformly integrable imply convergence of the moments:

$$\lim_{K \to \infty} \mathbb{E}[X_K] = \mathbb{E}[Y] = 1/(1 - \xi),$$

where the last part of the above the equation is just the moment of $Y$. The proof for the puts is analogous. □

C A Toy Example

To demonstrate the pricing formula with a toy example, assume $S_T$ has a uniform distribution on $(0, 1)$ and $r = 0$. (We are not claiming that the uniform distribution is a good model for the RND; this is just for illustration purposes.) Then for $x > 0$ and any $K < 1$:

$$\mathbb{P}(S_T - K > x|S_T > K) = \frac{\mathbb{P}(S_T > K + x)}{\mathbb{P}(S_T > K)} = \frac{1 - (K + x)}{1 - K} = 1 - \frac{x}{1 - K} = \left(1 - \frac{x}{1 - K}\right)^{(-1/-1)}$$

which is GP with $\beta(K) = 1 - K$, and $\xi = -1$. This example came from Coles (2001), page 77.

The risk neutral pricing from equation (6) gives:

$$C = \mathbb{E}[S_T - K|S_T > K]\mathbb{P}(S_T > K) = \mathbb{E}[(S_T - K)\mathbb{I}\{S_T > K\}] = \int_K^1 (x - K)dx = \frac{1}{2} - K + \frac{K^2}{2},$$

where $\mathbb{I}\{S_T > K\}$ is just the indicator function of the event $\{S_T > K\}$.

We also have $\beta(K) = 1 - K$, and $\xi = -1$. Now using equation (7), we get:

$$C^* = \frac{\beta(K)}{1 - \xi}\mathbb{P}(S_T > K) = \frac{1 - K}{1 - (-1)}(1 - K) = \frac{1}{2} - K + \frac{K^2}{2},$$

which is the exact result from the risk neutral valuation.
D An Intuitive Explanation for the Similarity of Results from Minimizations

The least squares and the absolute relative error minimization could produce similar parameter estimates when the fit is good. To see this, we use the intuitive arguments from Park and Stefanski (1998):

$$\min \sum_{i=0}^{n-1} (C_i^{\text{MKT}} - C_i^*)^2 = \min \sum_{i=0}^{n-1} (\log(C_i^{\text{MKT}}) - \log(C_i^*))^2$$

$$= \min \sum_{i=0}^{n-1} (\log(C_i^{\text{MKT}}/C_i^*))^2$$

$$\approx \min \sum_{i=0}^{n-1} (1 - C_i^*/C_i^{\text{MKT}})^2 \quad (\log(x) \approx x - 1, x \approx 1).$$
### Single Lognormal - Calls

<table>
<thead>
<tr>
<th>Strike</th>
<th>BSM</th>
<th>1063</th>
<th>1078</th>
<th>1098</th>
<th>1063</th>
<th>1078</th>
<th>1098</th>
<th>1063</th>
<th>1078</th>
<th>1098</th>
</tr>
</thead>
<tbody>
<tr>
<td>1100</td>
<td>3.76</td>
<td>3.82</td>
<td>3.79</td>
<td>3.77</td>
<td>0.16</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
</tr>
<tr>
<td>1110</td>
<td>2.90</td>
<td>2.98</td>
<td>2.95</td>
<td>2.92</td>
<td>0.14</td>
<td>0.15</td>
<td>0.15</td>
<td>0.16</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>1120</td>
<td>2.22</td>
<td>2.30</td>
<td>2.27</td>
<td>2.24</td>
<td>0.12</td>
<td>0.13</td>
<td>0.13</td>
<td>0.14</td>
<td>0.14</td>
<td>0.13</td>
</tr>
<tr>
<td>1130</td>
<td>1.69</td>
<td>1.76</td>
<td>1.73</td>
<td>1.71</td>
<td>0.10</td>
<td>0.11</td>
<td>0.11</td>
<td>0.13</td>
<td>0.12</td>
<td>0.12</td>
</tr>
<tr>
<td>1140</td>
<td>1.27</td>
<td>1.33</td>
<td>1.31</td>
<td>1.29</td>
<td>0.09</td>
<td>0.09</td>
<td>0.10</td>
<td>0.11</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>1150</td>
<td>0.95</td>
<td>1.00</td>
<td>0.98</td>
<td>0.97</td>
<td>0.08</td>
<td>0.08</td>
<td>0.08</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
</tr>
<tr>
<td>1160</td>
<td>0.70</td>
<td>0.74</td>
<td>0.73</td>
<td>0.72</td>
<td>0.07</td>
<td>0.07</td>
<td>0.07</td>
<td>0.08</td>
<td>0.07</td>
<td>0.07</td>
</tr>
<tr>
<td>1170</td>
<td>0.51</td>
<td>0.54</td>
<td>0.54</td>
<td>0.53</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>1180</td>
<td>0.37</td>
<td>0.39</td>
<td>0.39</td>
<td>0.38</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>1190</td>
<td>0.27</td>
<td>0.28</td>
<td>0.28</td>
<td>0.28</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>1200</td>
<td>0.19</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Table 3: Simulation Results for the single lognormal call.

### Single Lognormal - Puts

<table>
<thead>
<tr>
<th>Strike</th>
<th>BSM</th>
<th>938</th>
<th>925</th>
<th>908</th>
<th>938</th>
<th>925</th>
<th>908</th>
<th>938</th>
<th>925</th>
<th>908</th>
</tr>
</thead>
<tbody>
<tr>
<td>900</td>
<td>2.34</td>
<td>2.41</td>
<td>2.37</td>
<td>2.34</td>
<td>0.15</td>
<td>0.11</td>
<td>0.11</td>
<td>0.16</td>
<td>0.12</td>
<td>0.11</td>
</tr>
<tr>
<td>890</td>
<td>1.64</td>
<td>1.72</td>
<td>1.68</td>
<td>1.66</td>
<td>0.12</td>
<td>0.10</td>
<td>0.09</td>
<td>0.15</td>
<td>0.10</td>
<td>0.09</td>
</tr>
<tr>
<td>880</td>
<td>1.13</td>
<td>1.20</td>
<td>1.17</td>
<td>1.15</td>
<td>0.10</td>
<td>0.08</td>
<td>0.07</td>
<td>0.13</td>
<td>0.09</td>
<td>0.07</td>
</tr>
<tr>
<td>870</td>
<td>0.76</td>
<td>0.82</td>
<td>0.80</td>
<td>0.78</td>
<td>0.08</td>
<td>0.06</td>
<td>0.06</td>
<td>0.11</td>
<td>0.07</td>
<td>0.06</td>
</tr>
<tr>
<td>860</td>
<td>0.50</td>
<td>0.55</td>
<td>0.53</td>
<td>0.52</td>
<td>0.07</td>
<td>0.05</td>
<td>0.04</td>
<td>0.08</td>
<td>0.06</td>
<td>0.05</td>
</tr>
<tr>
<td>850</td>
<td>0.32</td>
<td>0.35</td>
<td>0.34</td>
<td>0.33</td>
<td>0.05</td>
<td>0.04</td>
<td>0.04</td>
<td>0.06</td>
<td>0.05</td>
<td>0.04</td>
</tr>
<tr>
<td>840</td>
<td>0.20</td>
<td>0.22</td>
<td>0.21</td>
<td>0.21</td>
<td>0.04</td>
<td>0.03</td>
<td>0.03</td>
<td>0.05</td>
<td>0.04</td>
<td>0.03</td>
</tr>
<tr>
<td>830</td>
<td>0.12</td>
<td>0.13</td>
<td>0.13</td>
<td>0.13</td>
<td>0.03</td>
<td>0.02</td>
<td>0.02</td>
<td>0.03</td>
<td>0.03</td>
<td>0.02</td>
</tr>
<tr>
<td>820</td>
<td>0.07</td>
<td>0.07</td>
<td>0.07</td>
<td>0.07</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>810</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.02</td>
<td>0.01</td>
<td>0.01</td>
<td>0.02</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>800</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table 4: Simulation Results for the single lognormal puts.

### E Tables

Tables (3) to (8) show the theoretical prices, GP based estimated mean prices, sample standard deviations, and the rmse's for the calls and puts respectively for 11 strikes. (Listing all strikes would have taken way too much space.)
### Generalized Beta (GB) - Calls

<table>
<thead>
<tr>
<th>Strike</th>
<th>GB Mean Price</th>
<th>SD Mean Price</th>
<th>RMSE Mean Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1150</td>
<td>1077</td>
<td>1096</td>
<td>1122</td>
</tr>
<tr>
<td>1160</td>
<td>2.72</td>
<td>2.80</td>
<td>2.77</td>
</tr>
<tr>
<td>1170</td>
<td>2.25</td>
<td>2.33</td>
<td>2.30</td>
</tr>
<tr>
<td>1180</td>
<td>1.86</td>
<td>1.93</td>
<td>1.90</td>
</tr>
<tr>
<td>1190</td>
<td>1.53</td>
<td>1.59</td>
<td>1.57</td>
</tr>
<tr>
<td>1200</td>
<td>1.26</td>
<td>1.31</td>
<td>1.30</td>
</tr>
<tr>
<td>1210</td>
<td>0.98</td>
<td>1.08</td>
<td>1.07</td>
</tr>
<tr>
<td>1220</td>
<td>0.70</td>
<td>0.72</td>
<td>0.72</td>
</tr>
<tr>
<td>1230</td>
<td>0.58</td>
<td>0.59</td>
<td>0.59</td>
</tr>
<tr>
<td>1240</td>
<td>0.48</td>
<td>0.48</td>
<td>0.48</td>
</tr>
</tbody>
</table>

Mean | SD | RMSE |
--- | --- | --- |
1150 | 0.18 | 0.15 | 0.20 | 0.19 | 0.15 |
1160 | 0.16 | 0.16 | 0.19 | 0.17 | 0.17 |
1170 | 0.15 | 0.15 | 0.17 | 0.16 | 0.15 |
1180 | 0.14 | 0.14 | 0.16 | 0.14 | 0.14 |
1190 | 0.12 | 0.12 | 0.14 | 0.13 | 0.13 |
1200 | 0.11 | 0.11 | 0.13 | 0.12 | 0.11 |
1210 | 0.10 | 0.10 | 0.11 | 0.11 | 0.10 |
1220 | 0.09 | 0.09 | 0.10 | 0.10 | 0.09 |
1230 | 0.08 | 0.08 | 0.08 | 0.08 | 0.08 |
1240 | 0.08 | 0.08 | 0.08 | 0.08 | 0.08 |
1250 | 0.07 | 0.07 | 0.07 | 0.07 | 0.07 |

Table 5: Simulation Results for the single Generalized Beta calls.

### Generalized Beta (GB) - Puts

<table>
<thead>
<tr>
<th>Strike</th>
<th>GB Mean Price</th>
<th>SD Mean Price</th>
<th>RMSE Mean Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>850</td>
<td>1.33</td>
<td>1.40</td>
<td>1.38</td>
</tr>
<tr>
<td>840</td>
<td>0.99</td>
<td>1.06</td>
<td>1.04</td>
</tr>
<tr>
<td>830</td>
<td>0.73</td>
<td>0.79</td>
<td>0.77</td>
</tr>
<tr>
<td>820</td>
<td>0.54</td>
<td>0.58</td>
<td>0.57</td>
</tr>
<tr>
<td>810</td>
<td>0.39</td>
<td>0.42</td>
<td>0.41</td>
</tr>
<tr>
<td>800</td>
<td>0.28</td>
<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>790</td>
<td>0.20</td>
<td>0.21</td>
<td>0.21</td>
</tr>
<tr>
<td>780</td>
<td>0.14</td>
<td>0.14</td>
<td>0.15</td>
</tr>
<tr>
<td>770</td>
<td>0.10</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>760</td>
<td>0.07</td>
<td>0.06</td>
<td>0.07</td>
</tr>
<tr>
<td>750</td>
<td>0.05</td>
<td>0.04</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Mean | SD | RMSE |
--- | --- | --- |
850 | 0.10 | 0.13 | 0.17 | 0.13 | 0.14 | 0.18 |
840 | 0.08 | 0.11 | 0.15 | 0.11 | 0.12 | 0.15 |
830 | 0.07 | 0.09 | 0.12 | 0.09 | 0.10 | 0.13 |
820 | 0.06 | 0.07 | 0.10 | 0.07 | 0.08 | 0.11 |
810 | 0.05 | 0.06 | 0.08 | 0.06 | 0.06 | 0.09 |
800 | 0.04 | 0.05 | 0.07 | 0.05 | 0.05 | 0.07 |
790 | 0.04 | 0.04 | 0.05 | 0.04 | 0.04 | 0.05 |
780 | 0.03 | 0.03 | 0.04 | 0.03 | 0.03 | 0.04 |
770 | 0.02 | 0.02 | 0.03 | 0.02 | 0.02 | 0.03 |
760 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
750 | 0.01 | 0.01 | 0.02 | 0.02 | 0.01 | 0.02 |

Table 6: Simulation Results for the single Generalized Beta puts.

### Mixture of Lognormals - Calls

<table>
<thead>
<tr>
<th>Strike</th>
<th>Mix Mean Price</th>
<th>Mix SD Mean Price</th>
<th>Mix RMSE Mean Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1125</td>
<td>1.47</td>
<td>1.65</td>
<td>1.55</td>
</tr>
<tr>
<td>1135</td>
<td>1.03</td>
<td>1.19</td>
<td>1.11</td>
</tr>
<tr>
<td>1145</td>
<td>0.71</td>
<td>0.84</td>
<td>0.78</td>
</tr>
<tr>
<td>1155</td>
<td>0.48</td>
<td>0.58</td>
<td>0.54</td>
</tr>
<tr>
<td>1165</td>
<td>0.32</td>
<td>0.39</td>
<td>0.36</td>
</tr>
<tr>
<td>1175</td>
<td>0.21</td>
<td>0.26</td>
<td>0.24</td>
</tr>
<tr>
<td>1185</td>
<td>0.14</td>
<td>0.17</td>
<td>0.15</td>
</tr>
<tr>
<td>1195</td>
<td>0.09</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>1205</td>
<td>0.05</td>
<td>0.06</td>
<td>0.05</td>
</tr>
<tr>
<td>1215</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>1225</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Mean | SD | RMSE |
--- | --- | --- |
1125 | 0.40 | 0.28 | 0.09 | 0.44 | 0.29 | 0.09 |
1135 | 0.34 | 0.24 | 0.07 | 0.37 | 0.25 | 0.07 |
1145 | 0.27 | 0.20 | 0.06 | 0.30 | 0.21 | 0.06 |
1155 | 0.22 | 0.16 | 0.05 | 0.24 | 0.17 | 0.05 |
1165 | 0.17 | 0.13 | 0.04 | 0.18 | 0.14 | 0.04 |
1175 | 0.12 | 0.10 | 0.03 | 0.13 | 0.10 | 0.03 |
1185 | 0.09 | 0.08 | 0.02 | 0.09 | 0.08 | 0.02 |
1195 | 0.06 | 0.06 | 0.02 | 0.06 | 0.06 | 0.02 |
1205 | 0.04 | 0.04 | 0.01 | 0.04 | 0.04 | 0.01 |
1215 | 0.03 | 0.03 | 0.01 | 0.03 | 0.03 | 0.01 |
1225 | 0.02 | 0.02 | 0.01 | 0.02 | 0.02 | 0.01 |

Table 7: Simulation Results for the mixture of lognormals calls.
Table 8: Simulation Results for the mixture of lognormals puts.

<table>
<thead>
<tr>
<th>Strike</th>
<th>Mean Price 897</th>
<th>SD 879</th>
<th>859</th>
<th>SD 897</th>
<th>RMSE 879</th>
<th>859</th>
<th>RMSE 897</th>
</tr>
</thead>
<tbody>
<tr>
<td>850</td>
<td>2.38 2.54 2.43</td>
<td>2.38 0.25 0.16</td>
<td>1.00 0.30 0.17</td>
<td>1.00 0.17 0.10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>840</td>
<td>1.68 1.86 1.75</td>
<td>1.69 0.21 0.14</td>
<td>0.08 0.28 0.16</td>
<td>0.08 0.16 0.08</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>830</td>
<td>1.15 1.33 1.24</td>
<td>1.18 0.18 0.12</td>
<td>0.07 0.26 0.14</td>
<td>0.07 0.14 0.07</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>820</td>
<td>0.77 0.93 0.85</td>
<td>0.80 0.15 0.10</td>
<td>0.05 0.22 0.13</td>
<td>0.05 0.13 0.06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>810</td>
<td>0.49 0.62 0.56</td>
<td>0.52 0.12 0.08</td>
<td>0.04 0.18 0.11</td>
<td>0.04 0.11 0.05</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>0.31 0.41 0.36</td>
<td>0.33 0.10 0.07</td>
<td>0.03 0.14 0.09</td>
<td>0.03 0.09 0.04</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>790</td>
<td>0.18 0.25 0.22</td>
<td>0.20 0.08 0.05</td>
<td>0.03 0.10 0.06</td>
<td>0.03 0.06 0.03</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>780</td>
<td>0.11 0.15 0.13</td>
<td>0.12 0.06 0.04</td>
<td>0.02 0.07 0.05</td>
<td>0.02 0.05 0.02</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>770</td>
<td>0.06 0.08 0.07</td>
<td>0.06 0.04 0.03</td>
<td>0.01 0.05 0.03</td>
<td>0.01 0.03 0.02</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>760</td>
<td>0.03 0.04 0.03</td>
<td>0.03 0.03 0.02</td>
<td>0.01 0.03 0.02</td>
<td>0.01 0.02 0.01</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>750</td>
<td>0.02 0.02 0.02</td>
<td>0.01 0.02 0.01</td>
<td>0.01 0.02 0.01</td>
<td>0.01 0.01 0.01</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

F Simulation Code

Note before running any of the code below, all the packages must be installed.

F.1 Lognormal

This section gives the simulation code for the lognormal risk neutral density.

```r
library(ismev)
### Define the constants
sample.size = 1000 * 10
n = 1000 * 20
t = 50/365
s0 = 1000
sigma = 0.20
y = 0.02
r = 0.03
mu.st = (r - y - (sigma^2)/2) * te
var.st = (sigma^2) * te
k0.80 = s0 * exp( qnorm(0.80)*sqrt(var.st) + mu.st) ### Theoretical 0.80 quantile of lognormal
k0.85 = s0 * exp( qnorm(0.85)*sqrt(var.st) + mu.st) ### Theoretical 0.85 quantile of lognormal
k0.90 = s0 * exp( qnorm(0.90)*sqrt(var.st) + mu.st) ### Theoretical 0.90 quantile of lognormal
k0.20 = s0 * exp( qnorm(0.20)*sqrt(var.st) + mu.st) ### Theoretical 0.20 quantile of lognormal
k0.15 = s0 * exp( qnorm(0.15)*sqrt(var.st) + mu.st) ### Theoretical 0.15 quantile of lognormal
k0.10 = s0 * exp( qnorm(0.10)*sqrt(var.st) + mu.st) ### Theoretical 0.10 quantile of lognormal
```

estimated.parameters = matrix(NA, n, 12)
colnames(estimated.parameters) = c("xi.call.80", "xi.call.85", "xi.call.90", "beta.call.80", "beta.call.85", "beta.call.90", "xi.put.20", "xi.put.15", "xi.put.10", "beta.put.20", "beta.put.15", "beta.put.10")
```

estimated.standarderrors = matrix(NA, n, 12)
colnames(estimated.standarderrors) = c("se.xi.call.80", "se.xi.call.85", "se.xi.call.90", "se.beta.call.80", "se.beta.call.85", "se.beta.call.90", "se.xi.put.20", "se.xi.put.15", "se.xi.put.10", "se.beta.put.20", "se.beta.put.15", "se.beta.put.10")
```
estimated.proportions = matrix(NA, n, 6)
colnames(estimated.proportions) = c("prop.greater.than.80", "prop.greater.than.85", "prop.greater.than.90", "prop.less.than.20", "prop.less.than.15", "prop.less.than.10")
convergence.message = matrix(NA, n, 6)
for (i in 1:n)
{
  x = s0 * exp(rnorm(sample.size, mean = mu.st, sd = sqrt(var.st)))
gpd.obj.80 = gpd.fit(x, k0.80, show=F)
gpd.obj.85 = gpd.fit(x, k0.85, show=F)
gpd.obj.90 = gpd.fit(x, k0.90, show=F)
gpd.obj.20 = gpd.fit(-x, -k0.20, show=F)
gpd.obj.15 = gpd.fit(-x, -k0.15, show=F)
gpd.obj.10 = gpd.fit(-x, -k0.10, show=F)
estimated.parameters[i,] = c(gpd.obj.80$mle[2], gpd.obj.85$mle[2], gpd.obj.90$mle[2], gpd.obj.80$mle[1], gpd.obj.85$mle[1], gpd.obj.90$mle[1], gpd.obj.20$mle[2], gpd.obj.15$mle[2], gpd.obj.10$mle[2], gpd.obj.20$mle[1], gpd.obj.15$mle[1], gpd.obj.10$mle[1])
estimated.standarderrors[i,] = c(gpd.obj.80$se[2], gpd.obj.85$se[2], gpd.obj.90$se[2], gpd.obj.80$se[1], gpd.obj.85$se[1], gpd.obj.90$se[1], gpd.obj.20$se[2], gpd.obj.15$se[2], gpd.obj.10$se[2], gpd.obj.20$se[1], gpd.obj.15$se[1], gpd.obj.10$se[1])
estimated.proportions[i,] = c(length(x[x > k0.80]) , length(x[x > k0.85]) , length(x[x > k0.90]) , length(x[x < k0.20]) , length(x[x < k0.15]) , length(x[x < k0.10]))/sample.size
convergence.message[i,] = c(gpd.obj.80$conv, gpd.obj.85$conv, gpd.obj.90$conv, gpd.obj.20$conv, gpd.obj.15$conv, gpd.obj.10$conv)
}

save.image("lognormal.RData")
load("lognormal.RData")

compute.rmse = function(estimates, true.value) {
  rmse = sqrt(mean((estimates - true.value)^2))
  rmse
}

bsm.option.price = function(r, te, S0, k, sigma, y) {
  d1 = (log(S0/k) + (r - y + (sigma^2)/2) * te) / (sigma * sqrt(te))
  d2 = d1 - sigma * sqrt(te)
  call.option.price = S0 * exp(-y*te) * pnorm(d1) - k * exp(-r*te) * pnorm(d2)
  put.option.price = k * exp(-r*te) * pnorm(-d2) - S0 * exp(-y*te) * pnorm(-d1)
  out = list(d1 = d1, d2 = d2, call = call.option.price, put = put.option.price)
  out
}

39
discount.factor = exp(-(r - y) * te)

### Identify the rows that did not converge

row.not.convrg.calls.80 = which(convergence.message[,1] != 0)
row.not.convrg.calls.85 = which(convergence.message[,2] != 0)
row.not.convrg.calls.90 = which(convergence.message[,3] != 0)
row.not.convrg.puts.20 = which(convergence.message[,4] != 0)
row.not.convrg.puts.15 = which(convergence.message[,5] != 0)
row.not.convrg.puts.10 = which(convergence.message[,6] != 0)

### Compute the initial puts and calls

c0.80 = (estimated.parameters[,4]/(1 - estimated.parameters[,1])) * discount.factor * estimated.proportions[,1]
c0.85 = (estimated.parameters[,5]/(1 - estimated.parameters[,2])) * discount.factor * estimated.proportions[,2]
c0.90 = (estimated.parameters[,6]/(1 - estimated.parameters[,3])) * discount.factor * estimated.proportions[,3]
p0.20 = (estimated.parameters[,10]/(1 - estimated.parameters[,7])) * discount.factor * estimated.proportions[,4]
p0.15 = (estimated.parameters[,11]/(1 - estimated.parameters[,8])) * discount.factor * estimated.proportions[,5]
p0.10 = (estimated.parameters[,12]/(1 - estimated.parameters[,9])) * discount.factor * estimated.proportions[,6]

### Compute rmse for the initial options

bsm.c0.80 = bsm.option.price(r, te, s0, k0.80, sigma, y)$call
bsm.c0.85 = bsm.option.price(r, te, s0, k0.85, sigma, y)$call
bsm.c0.90 = bsm.option.price(r, te, s0, k0.90, sigma, y)$call
bsm.p0.20 = bsm.option.price(r, te, s0, k0.20, sigma, y)$put
bsm.p0.15 = bsm.option.price(r, te, s0, k0.15, sigma, y)$put
bsm.p0.10 = bsm.option.price(r, te, s0, k0.10, sigma, y)$put

c0.80
summary(c0.80)
sd(c0.80)
c0.85
summary(c0.85)
sd(c0.85)
c0.90
summary(c0.90)
sd(c0.90)
p0.20
summary(p0.20)
sd(p0.20)
p0.15
summary(p0.15)
sd(p0.15)
p0.10
summary(p0.10)
sd(p0.10)

### Using the initial call values, estimate the call prices, and compare to BSM

### Note that k0.90 = 1098, k0.85 = 1078, and k0.80 = 1063

seq.k.calls = seq(from = 1100, to = 1200, by = 1)
```r
bsm.call.values = bsm.option.price(r, te, s0, seq.k.calls, sigma, y)$call

extreme.call.prices.80 = matrix(NA, n, length(seq.k.calls))
extreme.call.prices.85 = matrix(NA, n, length(seq.k.calls))
extreme.call.prices.90 = matrix(NA, n, length(seq.k.calls))

rmse.calls.80 = numeric(length(seq.k.calls))
rmse.calls.85 = numeric(length(seq.k.calls))
rmse.calls.90 = numeric(length(seq.k.calls))

###
### This loop computes the call prices
###
for (j in 1:length(seq.k.calls) ) {
  extreme.call.prices.80[,j] = c0.80 * ( (estimated.parameters[,1] / estimated.parameters[,4]) * (seq.k.calls[j] - k0.80) + 1 )^(1 - 1/estimated.parameters[,1])
  extreme.call.prices.85[,j] = c0.85 * ( (estimated.parameters[,2] / estimated.parameters[,5]) * (seq.k.calls[j] - k0.85) + 1 )^(1 - 1/estimated.parameters[,2])
  extreme.call.prices.90[,j] = c0.90 * ( (estimated.parameters[,3] / estimated.parameters[,6]) * (seq.k.calls[j] - k0.90) + 1 )^(1 - 1/estimated.parameters[,3])
}

###
### This loop computes the rmse
###
for (i in 1:length(seq.k.calls) ) {
  rmse.calls.80[i] = compute.rmse(extreme.call.prices.80[,i], bsm.call.values[i])
  rmse.calls.85[i] = compute.rmse(extreme.call.prices.85[,i], bsm.call.values[i])
  rmse.calls.90[i] = compute.rmse(extreme.call.prices.90[,i], bsm.call.values[i])
}

###
### Plot the rmse
###
y.upper.limit = max(max(rmse.calls.80), max(rmse.calls.85), max(rmse.calls.90))
xlim(height = 7, width = 7)
par(mar = c(5, 4, 4, 2) + 0.1) # default if c(5, 4, 4, 2) + 0.1
matplot(seq.k.calls, cbind(rmse.calls.80, rmse.calls.85, rmse.calls.90), ylim=c(0,y.upper.limit),
  xlab="Strike Values", ylab="RMSE Calls", col=c("green","blue","red"), cex.axis=1.5, cex.lab=1.5, type="l",
  lty=c(2,3,4), lwd = c(2.25,2.25,2.25))
abline(h=0, lty=2)
legend("topright", legend=c("0.80","0.85","0.90"), col=c("green","blue","red"), lty=c(2,3,4), lwd = c(3,3,3), bty="n", cex=1.5)

########################################################################
########################################################################
########################################################################

### Using the initial put values, estimate the put prices, and compare to BSM
###
### Note that k0.10 = 908, k0.15 = 925, and k0.20 = 938
###
seq.k.puts = seq(from = 900, to = 800, by = -1)
bsm.put.values = bsm.option.price(r, te, s0, seq.k.puts, sigma, y)$put

extreme.put.prices.20 = matrix(NA, n, length(seq.k.puts))
extreme.put.prices.15 = matrix(NA, n, length(seq.k.puts))
extreme.put.prices.10 = matrix(NA, n, length(seq.k.puts))

rmse.puts.20 = numeric(length(seq.k.puts))
rmse.puts.15 = numeric(length(seq.k.puts))
rmse.puts.10 = numeric(length(seq.k.puts))

###
### This loop computes the put prices
###
for (j in 1:length(seq.k.puts) ) {
  extreme.put.prices.20[,j] = p0.20 * ( (estimated.parameters[,7] / estimated.parameters[,10]) * (k0.20 - seq.k.puts[j]) + 1 )^(1 - 1/estimated.parameters[,7])
  extreme.put.prices.15[,j] = p0.15 * ( (estimated.parameters[,8] / estimated.parameters[,11]) * (k0.15 - seq.k.puts[j]) + 1 )^(1 - 1/estimated.parameters[,8])
  extreme.put.prices.10[,j] = p0.10 * ( (estimated.parameters[,9] / estimated.parameters[,12]) * (k0.10 - seq.k.puts[j]) + 1 )^(1 - 1/estimated.parameters[,9])
}
```
This loop computes the rmse

for (i in 1:length(seq.k.puts) )
{
  rmse.puts.20[i] = compute.rmse(extreme.put.prices.20[,i], bsm.put.values[i])
  rmse.puts.15[i] = compute.rmse(extreme.put.prices.15[,i], bsm.put.values[i])
  rmse.puts.10[i] = compute.rmse(extreme.put.prices.10[,i], bsm.put.values[i])
}

Plot the rmse

y.upper.limit = max(max(rmse.puts.20), max(rmse.puts.15), max(rmse.puts.10))

F.2 Generalized Beta

This section gives the simulation code for the Generalized Beta risk neutral density.
estimated.standarderrors = matrix(NA, n, 12)
colnames(estimated.standarderrors) = c("se.xi.call.80", "se.xi.call.85", "se.xi.call.90", "se.beta.call.80", "se.beta.call.85", "se.beta.call.90", "se.xi.put.20", "se.xi.put.15", "se.xi.put.10", "se.beta.put.20", "se.beta.put.15", "se.beta.put.10")

estimated.proportions = matrix(NA, n, 6)
colnames(estimated.proportions) = c("prop.greater.than.80", "prop.greater.than.85", "prop.greater.than.90", "prop.less.than.20", "prop.less.than.15", "prop.less.than.10")

convergence.message = matrix(NA, n, 6)

for (i in 1:n)
{
x = rgb2(sample.size, shape1 = shape1, scale = scale, shape2 = shape2, shape3 = shape3)

gpd.obj.80 = gpd.fit(x, k0.80, show=F)
gpd.obj.85 = gpd.fit(x, k0.85, show=F)
gpd.obj.90 = gpd.fit(x, k0.90, show=F)
gpd.obj.20 = gpd.fit(-x, -k0.20, show=F)
gpd.obj.15 = gpd.fit(-x, -k0.15, show=F)
gpd.obj.10 = gpd.fit(-x, -k0.10, show=F)

estimated.parameters[i,] = c(gpd.obj.80$mle[2], gpd.obj.85$mle[2], gpd.obj.90$mle[2], gpd.obj.20$mle[2], gpd.obj.15$mle[2], gpd.obj.10$mle[2], gpd.obj.80$mle[1], gpd.obj.85$mle[1], gpd.obj.90$mle[1], gpd.obj.20$mle[1], gpd.obj.15$mle[1], gpd.obj.10$mle[1])

estimated.standarderrors[i,] = c(gpd.obj.80$se[2], gpd.obj.85$se[2], gpd.obj.90$se[2], gpd.obj.20$se[2], gpd.obj.15$se[2], gpd.obj.10$se[2], gpd.obj.80$se[1], gpd.obj.85$se[1], gpd.obj.90$se[1], gpd.obj.20$se[1], gpd.obj.15$se[1], gpd.obj.10$se[1])

estimated.proportions[i,] = c(length(x[x > k0.80]), length(x[x > k0.85]), length(x[x > k0.90]), length(x[x < k0.20]), length(x[x < k0.15]), length(x[x < k0.10]))/sample.size

convergence.message[i,] = c(gpd.obj.80$conv, gpd.obj.85$conv, gpd.obj.90$conv, gpd.obj.20$conv, gpd.obj.15$conv, gpd.obj.10$conv)
}

save.image("gb.RData")
load("gb.RData")

####################################################################################################################################

#### Simple Function to compute rmse
####
#### compute.rmse = function(estimates, true.value)
#### {
#### rmse = sqrt(mean((estimates - true.value)^2))
#### rmse
#### }

####################################################################################################################################

#### GB based Option Pricing Formula.
####
#### Note all units should be in years
####
#### Note shape1 = a, scale = b, shape2 = p and shape3 = q
####
#### gb.option.price = function(r, te, s0, k, a, b, p, q)
#### {
#### prob.1 = pgb2(x, shape1 = a, scale = b, shape2 = (p + 1/a) , shape3 = (q - 1/a) )
#### prob.2 = pgb2(x, shape1 = a, scale = b, shape2 = p , shape3 = q )
####
call.option.price = s0 * ( 1 - prob.1) - k * exp(-r*te) * ( 1 - prob.2)
put.option.price = call.option.price - s0 + k * exp(-r*te)

out = list(prob.1 = prob.1, prob.2 = prob.2, call = call.option.price, put = put.option.price)
out
}
\[ r = 0.03 \]
\[ t_e = \frac{50}{365} \]
\[ \text{discount.factor} = \exp(-r \times t_e) \]
\[ s_0 = \frac{(\text{discount.factor} \times \text{scale} \times \text{beta}(\text{shape2} + 1/\text{shape1}, \text{shape3} - 1/\text{shape1}))}{\text{beta}(\text{shape2}, \text{shape3})} \]

### Identify the rows that did not converge

\[
\text{row.not.convrg.calls.80} = \text{which}(\text{convergence.message[,1]} != 0) \\
\text{row.not.convrg.calls.85} = \text{which}(\text{convergence.message[,2]} != 0) \\
\text{row.not.convrg.calls.90} = \text{which}(\text{convergence.message[,3]} != 0) \\
\text{row.not.convrg.puts.20} = \text{which}(\text{convergence.message[,4]} != 0) \\
\text{row.not.convrg.puts.15} = \text{which}(\text{convergence.message[,5]} != 0) \\
\text{row.not.convrg.puts.10} = \text{which}(\text{convergence.message[,6]} != 0) 
\]

### Compute the initial puts and calls

\[
c_{0.80} = \frac{(\text{estimated.parameters[,4]} / (1 - \text{estimated.parameters[,1]})) \times \text{discount.factor} \times \text{estimated.proportions[,1]}}{\text{gb.option.price}(r, t_e, s_0, k_{0.80}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3})} \]
\[
c_{0.85} = \frac{(\text{estimated.parameters[,5]} / (1 - \text{estimated.parameters[,2]})) \times \text{discount.factor} \times \text{estimated.proportions[,2]}}{\text{gb.option.price}(r, t_e, s_0, k_{0.85}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3})} \]
\[
c_{0.90} = \frac{(\text{estimated.parameters[,6]} / (1 - \text{estimated.parameters[,3]})) \times \text{discount.factor} \times \text{estimated.proportions[,3]}}{\text{gb.option.price}(r, t_e, s_0, k_{0.90}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3})} \]
\[
p_{0.20} = \frac{(\text{estimated.parameters[,10]} / (1 - \text{estimated.parameters[,7]})) \times \text{discount.factor} \times \text{estimated.proportions[,4]}}{\text{gb.option.price}(r, t_e, s_0, k_{0.20}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3})} \]
\[
p_{0.15} = \frac{(\text{estimated.parameters[,11]} / (1 - \text{estimated.parameters[,8]})) \times \text{discount.factor} \times \text{estimated.proportions[,5]}}{\text{gb.option.price}(r, t_e, s_0, k_{0.15}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3})} \]
\[
p_{0.10} = \frac{(\text{estimated.parameters[,12]} / (1 - \text{estimated.parameters[,9]})) \times \text{discount.factor} \times \text{estimated.proportions[,6]}}{\text{gb.option.price}(r, t_e, s_0, k_{0.10}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3})} \]

### Compute rmse for the initial options

\[
\text{gb.c0.80} = \text{gb.option.price}(r, t_e, s_0, k_{0.80}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3}) \text{ } \text{call} \\
\text{gb.c0.85} = \text{gb.option.price}(r, t_e, s_0, k_{0.85}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3}) \text{ } \text{call} \\
\text{gb.c0.90} = \text{gb.option.price}(r, t_e, s_0, k_{0.90}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3}) \text{ } \text{call} \\
\text{gb.p0.20} = \text{gb.option.price}(r, t_e, s_0, k_{0.20}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3}) \text{ } \text{put} \\
\text{gb.p0.15} = \text{gb.option.price}(r, t_e, s_0, k_{0.15}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3}) \text{ } \text{put} \\
\text{gb.p0.10} = \text{gb.option.price}(r, t_e, s_0, k_{0.10}, \text{shape1}, \text{scale}, \text{shape2}, \text{shape3}) \text{ } \text{put} 
\]

\[
\text{compute.rmse}(c_{0.80}, \text{gb.c0.80}) \\
\text{gb.c0.80} \text{ summary}(c_{0.80}) \text{ sd}(c_{0.80}) \\
\text{compute.rmse}(c_{0.85}, \text{gb.c0.85}) \\
\text{gb.c0.85} \text{ summary}(c_{0.85}) \text{ sd}(c_{0.85}) \\
\text{compute.rmse}(c_{0.90}, \text{gb.c0.90}) \\
\text{gb.c0.90} \text{ summary}(c_{0.90}) \text{ sd}(c_{0.90}) \\
\text{compute.rmse}(p_{0.20}, \text{gb.p0.20}) \\
\text{gb.p0.20} \text{ summary}(p_{0.20}) \text{ sd}(p_{0.20}) \\
\text{compute.rmse}(p_{0.15}, \text{gb.p0.15}) \\
\text{gb.p0.15} \text{ summary}(p_{0.15}) \text{ sd}(p_{0.15}) \\
\text{compute.rmse}(p_{0.10}, \text{gb.p0.10}) \\
\text{gb.p0.10} \text{ summary}(p_{0.10}) \text{ sd}(p_{0.10}) 
\]

44
### Using the initial call values, estimate the call prices, and compare to GB Based

### Note that \(k_{0.90} = 1121.826\), \(k_{0.85} = 1096.563\), and \(k_{0.80} = 1077.284\)

```r
seq.k.calls = seq(from = 1150, to = 1250, by = 1)
gb.call.values = gb.option.price(r, te, s0, seq.k.calls, shape1, scale, shape2, shape3)$call

extreme.call.prices.80 = matrix(NA, n, length(seq.k.calls))
extreme.call.prices.85 = matrix(NA, n, length(seq.k.calls))
extreme.call.prices.90 = matrix(NA, n, length(seq.k.calls))
```

```r
rmse.calls.80 = numeric(length(seq.k.calls))
rmse.calls.85 = numeric(length(seq.k.calls))
rmse.calls.90 = numeric(length(seq.k.calls))
```

### This loop computes the call prices

```r
for (j in 1:length(seq.k.calls) )
{
  extreme.call.prices.80[,j] = c0.80 * ( (estimated.parameters[,1] / estimated.parameters[,4]) * (seq.k.calls[j] - k0.80) + 1 )^{(1 - 1/estimated.parameters[,1])}
  extreme.call.prices.85[,j] = c0.85 * ( (estimated.parameters[,2] / estimated.parameters[,5]) * (seq.k.calls[j] - k0.85) + 1 )^{(1 - 1/estimated.parameters[,2])}
  extreme.call.prices.90[,j] = c0.90 * ( (estimated.parameters[,3] / estimated.parameters[,6]) * (seq.k.calls[j] - k0.90) + 1 )^{(1 - 1/estimated.parameters[,3])}
}
```

### This loop computes the rmse

```r
for (i in 1:length(seq.k.calls) )
{
  rmse.calls.80[i] = compute.rmse(extreme.call.prices.80[,i], gb.call.values[i])
  rmse.calls.85[i] = compute.rmse(extreme.call.prices.85[,i], gb.call.values[i])
  rmse.calls.90[i] = compute.rmse(extreme.call.prices.90[,i], gb.call.values[i])
}
```

### Plot the rmse

```r
y.upper.limit = max(max(rmse.calls.80), max(rmse.calls.85), max(rmse.calls.90))
x11(height = 7, width = 7)
pastcol = c(5, 6, 4, 3) + 0.15  # default if c(5, 6, 4, 3) + 0.1
matplot(seq.k.calls, cbind(rmse.calls.80, rmse.calls.85, rmse.calls.90), ylim=c(0,y.upper.limit),
  xlab="Strike Values", ylab="RMSE Calls", col=c("green","blue","red"), cex.axis=1.5, cex.lab=1.5, type="l",
  lty=c(2,3,4), lwd = c(2.25,2.25,2.25))
abline(h=0, lty=2)
legend("topright", legend=c("0.80","0.85","0.90"), col=c("green","blue","red"), lty=c(2,3,4), lwd = c(3,3,3), bty="n", cex=1.5)
```

### Using the initial put values, estimate the put prices, and compare to GB

### Note that \(k_{0.10} = 891.404\), \(k_{0.15} = 911.94\), and \(k_{0.20} = 928.2605\)

```r
seq.k.puts = seq(from = 850, to = 750, by = -1)
gb.put.values = gb.option.price(r, te, s0, seq.k.puts, shape1, scale, shape2, shape3)$put
```

```r
extreme.put.prices.20 = matrix(NA, n, length(seq.k.puts))
extreme.put.prices.15 = matrix(NA, n, length(seq.k.puts))
extreme.put.prices.10 = matrix(NA, n, length(seq.k.puts))
```

```r
rmse.puts.20 = numeric(length(seq.k.puts))
rmse.puts.15 = numeric(length(seq.k.puts))
rmse.puts.10 = numeric(length(seq.k.puts))
```
This loop computes the put prices

```r
for (j in 1:length(seq.k.puts) )
{
  extreme.put.prices.20[,j] = p0.20 * ( (estimated.parameters[,7] / estimated.parameters[,10]) * (k0.20 - seq.k.puts[j]) + 1 )^(1 - 1/estimated.parameters[,7])
  extreme.put.prices.15[,j] = p0.15 * ( (estimated.parameters[,8] / estimated.parameters[,11]) * (k0.15 - seq.k.puts[j]) + 1 )^(1 - 1/estimated.parameters[,8])
  extreme.put.prices.10[,j] = p0.10 * ( (estimated.parameters[,9] / estimated.parameters[,12]) * (k0.10 - seq.k.puts[j]) + 1 )^(1 - 1/estimated.parameters[,9])
}
```

This loop computes the rmse

```r
for (i in 1:length(seq.k.puts) )
{
  rmse.puts.20[i] = compute.rmse(extreme.put.prices.20[,i], gb.put.values[i])
  rmse.puts.15[i] = compute.rmse(extreme.put.prices.15[,i], gb.put.values[i])
  rmse.puts.10[i] = compute.rmse(extreme.put.prices.10[,i], gb.put.values[i])
}
```

Plot the rmse

```r
y.upper.limit = max(max(rmse.puts.20), max(rmse.puts.15), max(rmse.puts.10))
xlim(height = 7, width = 7)
par(mar = c(5, 5, 4, 3) + 0.15) # default if c(5, 4, 4, 2) + 0.1
matplot(seq.k.puts, cbind(rmse.puts.20, rmse.puts.15, rmse.puts.10), ylim=c(0,y.upper.limit),
xlab="Strike Values", ylab="RMSE Puts", col=c("green","blue","red"), cex.axis=1.5, cex.lab=1.5, type="l",
lty=c(2,3,4), lwd = c(2.25,2.25,2.25))
abline(h=0, lty=2)
legend("topleft", legend=c("0.20","0.15","0.10"), col=c("green","blue","red"), lty=c(2,3,4), lwd = c(3,3,3), bty="n", cex=1.5)
```

### F.3 Mixture of Lognormals

This section gives the simulation code for the mixture of lognormals risk neutral density.

```r
def h(xq, q.prob, meanlog.1 = 1, meanlog.2 = 1, sdlog.1 = 0.1, sdlog.2 = 0.1, alpha.1 = 0.5, alpha.2 = 0.50)
{
  A = plnorm(xq, meanlog = meanlog.1, sdlog = sdlog.1)
  B = plnorm(xq, meanlog = meanlog.2, sdlog = sdlog.2)
  out = q.prob - (alpha.1 * A) - (alpha.2 * B)
  out
}
```
library(ismev)

### Define the constants

sample.size.total = 1000 * 10
n = 1000 * 20
alpha.1 = 0.40
alpha.2 = 1 - alpha.1
sample.size.1 = sample.size.total * alpha.1
sample.size.2 = sample.size.total - sample.size.1
meanlog.1 = 6.80
meanlog.2 = 6.95
sdlog.1 = 0.065
sdlog.2 = 0.055

### Find the theoretical quantiles of the mixture

k0.10 = uniroot(h, interval = c(1,1300), q.prob = 0.1, meanlog.1 = meanlog.1,
meanlog.2 = meanlog.2, sdlog.1 = sdlog.1, sdlog.2 =sdlog.2, alpha.1 = alpha.1, alpha.2 = alpha.2)$root
k0.15 = uniroot(h, interval = c(1,1300), q.prob = 0.15, meanlog.1 = meanlog.1,
meanlog.2 = meanlog.2, sdlog.1 = sdlog.1, sdlog.2 =sdlog.2, alpha.1 = alpha.1, alpha.2 = alpha.2)$root
k0.20 = uniroot(h, interval = c(1,1300), q.prob = 0.20, meanlog.1 = meanlog.1,
meanlog.2 = meanlog.2, sdlog.1 = sdlog.1, sdlog.2 =sdlog.2, alpha.1 = alpha.1, alpha.2 = alpha.2)$root
k0.80 = uniroot(h, interval = c(1,1300), q.prob = 0.80, meanlog.1 = meanlog.1,
meanlog.2 = meanlog.2, sdlog.1 = sdlog.1, sdlog.2 =sdlog.2, alpha.1 = alpha.1, alpha.2 = alpha.2)$root
k0.85 = uniroot(h, interval = c(1,1300), q.prob = 0.85, meanlog.1 = meanlog.1,
meanlog.2 = meanlog.2, sdlog.1 = sdlog.1, sdlog.2 =sdlog.2, alpha.1 = alpha.1, alpha.2 = alpha.2)$root
k0.90 = uniroot(h, interval = c(1,1300), q.prob = 0.90, meanlog.1 = meanlog.1,
meanlog.2 = meanlog.2, sdlog.1 = sdlog.1, sdlog.2 =sdlog.2, alpha.1 = alpha.1, alpha.2 = alpha.2)$root

estimated.parameters = matrix(NA, n, 12)
colnames(estimated.parameters) = c("xi.call.80", "xi.call.85", "xi.call.90", "beta.call.80", "beta.call.85", "beta.call.90",
"xi.put.20", "xi.put.15", "xi.put.10", "beta.put.20", "beta.put.15", "beta.put.10")
estimated.standarderrors = matrix(NA, n, 12)
colnames(estimated.standarderrors) = c("se.xi.call.80", "se.xi.call.85", "se.xi.call.90", "se.beta.call.80", "se.beta.call.85", "se.beta.call.90",
"se.xi.put.20", "se.xi.put.15", "se.xi.put.10", "se.beta.put.20", "se.beta.put.15", "se.beta.put.10")
estimated.proportions = matrix(NA, n, 6)
colnames(estimated.proportions) = c("prop.greater.than.80", "prop.greater.than.85", "prop.greater.than.90",
"prop.less.than.20", "prop.less.than.15", "prop.less.than.10")
convergence.message = matrix(NA, n, 6)

for (i in 1:n) {
  x = c(rlnorm(sample.size.1, meanlog = meanlog.1, sdlog = sdlog.1), rlnorm(sample.size.2, meanlog = meanlog.2, sdlog = sdlog.2) )
  gpd.obj.80 = gpd.fit(x, k0.80, show=F)
  gpd.obj.85 = gpd.fit(x, k0.85, show=F)
  gpd.obj.90 = gpd.fit(x, k0.90, show=F)
  gpd.obj.20 = gpd.fit(-x, -k0.20, show=F)
  gpd.obj.15 = gpd.fit(-x, -k0.15, show=F)
  gpd.obj.10 = gpd.fit(-x, -k0.10, show=F)

  estimated.parameters[i,] = c(gpd.obj.80$mle[2], gpd.obj.85$mle[2], gpd.obj.90$mle[2],
  gpd.obj.80$mle[1], gpd.obj.85$mle[1], gpd.obj.90$mle[1],
  gpd.obj.20$mle[2], gpd.obj.15$mle[2], gpd.obj.10$mle[2],
  gpd.obj.20$mle[1], gpd.obj.15$mle[1], gpd.obj.10$mle[1])

  estimated.standarderrors[i,] = c(gpd.obj.80$se[2], gpd.obj.85$se[2], gpd.obj.90$se[2],
  gpd.obj.80$se[1], gpd.obj.85$se[1], gpd.obj.90$se[1],
  gpd.obj.20$se[2], gpd.obj.15$se[2], gpd.obj.10$se[2],
  gpd.obj.20$se[1], gpd.obj.15$se[1], gpd.obj.10$se[1])

  estimated.proportions[i,] = c(length(x > k0.80), length(x > k0.85), length(x > k0.90),
  length(x < k0.20), length(x < k0.15), length(x < k0.10))/sample.size.total

  convergence.message[i,] = c(gpd.obj.80$conv, gpd.obj.85$conv, gpd.obj.90$conv, gpd.obj.20$conv, gpd.obj.15$conv, gpd.obj.10$conv)
}


47

####################################################################################################################################
save.image("mixture.RData")
load("mixture.RData")

### Simple Function to compute rmse
###
compute.rmse = function(estimates, true.value)
{
  rmse = sqrt(mean((estimates - true.value)^2))
  rmse
}

### Mixture Option Pricing Formula.
###
mixture.option.price = function(r, te, s0, k, alpha.1, alpha.2, meanlog.1, meanlog.2, sdlog.1, sdlog.2)
{
  discout.factor = exp(-r * te)
  
  ### First Component
  u1 = (log(k) - meanlog.1)/sdlog.1
tmp1 = exp(meanlog.1 + (0.5)*(sdlog.1^2)) * (1 - pnorm(u1 - sdlog.1)) - k * (1 - pnorm(u1))
c1 = discout.factor * tmp1
  u2 = (log(k) - meanlog.2)/sdlog.2
tmp2 = exp(meanlog.2 + (0.5)*(sdlog.2^2)) * (1 - pnorm(u2 - sdlog.2)) - k * (1 - pnorm(u2))
c2 = discout.factor * tmp2
  call.option.price = alpha.1 * c1 + alpha.2 * c2
  put.option.price = call.option.price - s0 + k * discout.factor

  out = list(call = call.option.price, put = put.option.price)
  out
}

r = 0.03
te = 50/365
discount.factor = exp(-r * te)
s0 = discount.factor * (alpha.1 * exp(meanlog.1 + 0.5 * sdlog.1^2) + alpha.2 * exp(meanlog.2 + 0.5 * sdlog.2^2))

### Identify the rows that did not converge
###
row.not.convrg.calls.80 = which(convergence.message[,1] != 0)
row.not.convrg.calls.85 = which(convergence.message[,2] != 0)
row.not.convrg.calls.90 = which(convergence.message[,3] != 0)
row.not.convrg.puts.20 = which(convergence.message[,4] != 0)
row.not.convrg.puts.15 = which(convergence.message[,5] != 0)
row.not.convrg.puts.10 = which(convergence.message[,6] != 0)

### Compute the initial puts and calls
###
c0.80 = (estimated.parameters[,4]/(1 - estimated.parameters[,1])) * discount.factor * estimated.proportions[,1]
c0.85 = (estimated.parameters[,5]/(1 - estimated.parameters[,2])) * discount.factor * estimated.proportions[,2]
c0.90 = (estimated.parameters[,6]/(1 - estimated.parameters[,3])) * discount.factor * estimated.proportions[,3]
p0.20 = (estimated.parameters[,10]/(1 - estimated.parameters[,7])) * discount.factor * estimated.proportions[,4]
\[ p_{0.15} = \left( \frac{\text{estimated.parameters}[11]}{1 - \text{estimated.parameters}[8]} \right) \times \text{discount.factor} \times \text{estimated.proportions}[5] \]

\[ p_{0.10} = \left( \frac{\text{estimated.parameters}[12]}{1 - \text{estimated.parameters}[9]} \right) \times \text{discount.factor} \times \text{estimated.proportions}[6] \]

### Compute rmse for the initial options

```r
mix.c0.80 = \text{mixture.option.price}(r, te, s0, k0.80, \alpha.1, \alpha.2, \text{meanlog.1}, \text{meanlog.2}, \text{sdlog.1}, \text{sdlog.2})\text{call}
mix.c0.85 = \text{mixture.option.price}(r, te, s0, k0.85, \alpha.1, \alpha.2, \text{meanlog.1}, \text{meanlog.2}, \text{sdlog.1}, \text{sdlog.2})\text{call}
mix.c0.90 = \text{mixture.option.price}(r, te, s0, k0.90, \alpha.1, \alpha.2, \text{meanlog.1}, \text{meanlog.2}, \text{sdlog.1}, \text{sdlog.2})\text{call}
mix.p0.20 = \text{mixture.option.price}(r, te, s0, k0.20, \alpha.1, \alpha.2, \text{meanlog.1}, \text{meanlog.2}, \text{sdlog.1}, \text{sdlog.2})\text{put}
mix.p0.15 = \text{mixture.option.price}(r, te, s0, k0.15, \alpha.1, \alpha.2, \text{meanlog.1}, \text{meanlog.2}, \text{sdlog.1}, \text{sdlog.2})\text{put}
mix.p0.10 = \text{mixture.option.price}(r, te, s0, k0.10, \alpha.1, \alpha.2, \text{meanlog.1}, \text{meanlog.2}, \text{sdlog.1}, \text{sdlog.2})\text{put}
```

```r
compute.rmse(c0.80, mix.c0.80)
mix.c0.80
summary(c0.80)
sd(c0.80)
```

```r
compute.rmse(c0.85, mix.c0.85)
mix.c0.85
summary(c0.85)
sd(c0.85)
```

```r
compute.rmse(c0.90, mix.c0.90)
mix.c0.90
summary(c0.90)
sd(c0.90)
```

```r
compute.rmse(p0.20, mix.p0.20)
mix.p0.20
summary(p0.20)
sd(p0.20)
```

```r
compute.rmse(p0.15, mix.p0.15)
mix.p0.15
summary(p0.15)
sd(p0.15)
```

```r
compute.rmse(p0.10, mix.p0.10)
mix.p0.10
summary(p0.10)
sd(p0.10)
```

```
Note that k0.90 = 1100.299, k0.85 = 1082.821, and k0.80 = 1068.557
```

```r
seq.k.calls = seq(from = 1125, to = 1225, by = 1)
mix.call.values = \text{mixture.option.price}(r, te, s0, seq.k.calls, \alpha.1, \alpha.2, \text{meanlog.1}, \text{meanlog.2}, \text{sdlog.1}, \text{sdlog.2})\text{call}
```

```r
extreme.call.prices.80 = \text{matrix}(NA, n, \text{length(seq.k.calls)})
extreme.call.prices.85 = \text{matrix}(NA, n, \text{length(seq.k.calls)})
extreme.call.prices.90 = \text{matrix}(NA, n, \text{length(seq.k.calls)})
```

```r
rmse.calls.80 = \text{numeric}(\text{length(seq.k.calls)})
rmse.calls.85 = \text{numeric}(\text{length(seq.k.calls)})
rmse.calls.90 = \text{numeric}(\text{length(seq.k.calls)})
```

### Using the initial call values, estimate the call prices, and compare to mixture of lognormals

```r
for (j in 1:length(seq.k.calls) )
{
        extreme.call.prices.80[, j] = c0.80 \times \left( \frac{\text{estimated.parameters}[1]}{\text{estimated.parameters}[4]} \right) \times \left( \text{seq.k.calls[j]} - k0.80 \right) + 1 \right)^{1 - \frac{1}{\text{estimated.parameters}[1]}}
        extreme.call.prices.85[, j] = c0.85 \times \left( \frac{\text{estimated.parameters}[2]}{\text{estimated.parameters}[5]} \right) \times \left( \text{seq.k.calls[j]} - k0.85 \right) + 1 \right)^{1 - \frac{1}{\text{estimated.parameters}[2]}}
        extreme.call.prices.90[, j] = c0.90 \times \left( \frac{\text{estimated.parameters}[3]}{\text{estimated.parameters}[6]} \right) \times \left( \text{seq.k.calls[j]} - k0.90 \right) + 1 \right)^{1 - \frac{1}{\text{estimated.parameters}[3]}}
}
```
This loop computes the rmse

```r
for (i in 1:length(seq.k.calls) )
{
  rmse.calls.80[i] = compute.rmse(extreme.call.prices.80[,i], mix.call.values[i])
  rmse.calls.85[i] = compute.rmse(extreme.call.prices.85[,i], mix.call.values[i])
  rmse.calls.90[i] = compute.rmse(extreme.call.prices.90[,i], mix.call.values[i])
}
```

Plot the rmse

```r
y.upper.limit = max(max(rmse.calls.80), max(rmse.calls.85), max(rmse.calls.90))
```

```r
x11(height = 7, width = 7)
par(mar = c(5, 5, 4, 3) + 0.1)
matplot(seq.k.calls, cbind(rmse.calls.80, rmse.calls.85, rmse.calls.90), ylim=c(0,y.upper.limit),
xlab="Strike Values", ylab="RMSE Calls", col=c("green","blue","red"), cex.axis=1.5, cex.lab=1.5, type="l",
lty=c(2,3,4), lwd = c(2.25,2.25,2.25))
abline(h=0, lty=2)
legend("topright", legend=c("0.80","0.85","0.90"), col=c("green","blue","red"), lty=c(2,3,4), lwd = c(3,3,3), bty="n", cex=1.5)
```

Using the initial put values, estimate the put prices, and compare to mixture of lognormals

```r
seq.k.puts = seq(from = 850, to = 750, by = -1)
mix.put.values = mixture.option.price(r, te, s0, seq.k.puts, alpha.1, alpha.2, meanlog.1, meanlog.2, sdlog.1, sdlog.2)$put
```

```r
extreme.put.prices.20 = matrix(NA, n, length(seq.k.puts))
extreme.put.prices.15 = matrix(NA, n, length(seq.k.puts))
extreme.put.prices.10 = matrix(NA, n, length(seq.k.puts))
```

```r
for (j in 1:length(seq.k.puts) )
{
  extreme.put.prices.20[,j] = p0.20 * ( (estimated.parameters[,7] / estimated.parameters[,10]) * (k0.20 - seq.k.puts[j]) + 1 )^(1 - 1/estimated.parameters[,7])
  extreme.put.prices.15[,j] = p0.15 * ( (estimated.parameters[,8] / estimated.parameters[,11]) * (k0.15 - seq.k.puts[j]) + 1 )^(1 - 1/estimated.parameters[,8])
  extreme.put.prices.10[,j] = p0.10 * ( (estimated.parameters[,9] / estimated.parameters[,12]) * (k0.10 - seq.k.puts[j]) + 1 )^(1 - 1/estimated.parameters[,9])
}
```

Plot the rmse

```r
y.upper.limit = max(max(rmse.puts.20), max(rmse.puts.15), max(rmse.puts.10))
```

```r
x11(height = 7, width = 7)
par(mar = c(5, 5, 4, 3) + 0.15) # default if c(5, 4, 4, 2) + 0.1
matplot(seq.k.puts, cbind(rmse.puts.20, rmse.puts.15, rmse.puts.10), ylim=c(0,y.upper.limit),
xlab="Strike Values", ylab="RMSE Puts", col=c("green","blue","red"), cex.axis=1.5, cex.lab=1.5, type="l",
lty=c(2,3,4), lwd = c(2.25,2.25,2.25))
abline(h=0, lty=2)
legend("topright", legend=c("0.20","0.15","0.10"), col=c("green","blue","red"), lty=c(2,3,4), lwd = c(3,3,3), bty="n", cex=1.5)
```
abline(h=0, lty=2)
legend("topleft", legend=c("0.20","0.15","0.10"), col=c("green","blue","red"), lty=c(2,3,4), lwd = c(3,3,3), bty="n", cex=1.5)

###
### Extraction Saved
###
save.image("mix-extracted.RData")
G

S&P 500 Analysis Code

Copying and pasting the code below will not reproduce the results in section (7). The original data is needed. However, we have this code as a reference.

load("large-opdata.RData")

###
### This is the objective function for minimizing the relative error.
###

objective.function = function(theta, c0, k0, call.prices, strikes)
{
  xi = theta[1]
  beta = theta[2]
  sample.size = length(call.prices)
  check.terms = 1 + (xi/beta)*(strikes - k0)
  presence.of.neg.indicator = sum(check.terms <= 0)
  if ( presence.of.neg.indicator > 0 ) {out = 10^7} else {
    y = call.prices
    x = (c0) * ((check.terms)^(1 - 1/xi))
    out = sum(abs(1 - x/y))
  }
  out
}

###
### This function extracts the tail shape for puts but it is based on minimizing the relative error.
###

extract rnd.tail.shape.puts = function(op.data, days, no.days, lower.q, min.open.interest , min.bid.price , initial.values = c(-0.20,100), max.row = 100)
{
  xi.estimates = numeric(no.days)
  xi.se = numeric(no.days)
  beta.estimates = numeric(no.days)
  beta.se = numeric(no.days)
  p0.vector = numeric(no.days)
  k0.vector = numeric(no.days)
  put.matrix = matrix(NA, max.row, no.days) ### columns correspond to the day. This matrix hold the market put values. It will include the first put.
  predicted.put.matrix = matrix(NA, max.row, no.days) ### This matrix holds the predicted put values from the model.
  strike.matrix = matrix(NA, max.row, no.days) ### This matrix holds the strike values.
  put.data.only = subset(op.data, cp_flag == "P")
  put.data.only = transform(put.data.only, strike_price = strike_price/1000)
  put.data.only = subset(put.data.only, open_interest > min.open.interest)
  put.data.only = subset(put.data.only, best_bid > min.bid.price)
  dim.sizes = numeric(no.days) ### This shows the number of puts with unique strikes per day
  end.index.sizes = numeric(no.days) ### This shows the number of puts used for tail index estimation. It includes the initial put too.
  did.it.converge = numeric(no.days)
  initial.xi = initial.values[1]
  initial.beta = initial.values[2]

  for (i in 1:no.days)
  {
    temp.data = subset(put.data.only, date == days[i] , select = c(strike_price, option_price ) )
    sorted.temp.data = unique(temp.data[order(temp.data$strike_price),])
    len = dim(sorted.temp.data)[1]
    dim.sizes[i] = len
    end.index.sizes[i] = ceiling(len * lower.q)
put.prices = sorted.temp.data$option_price[1:end.index.sizes[i]]
strikes = sorted.temp.data$strike_price[1:end.index.sizes[i]]

put.matrix[,i] = c(put.prices, rep(NA, max.row - end.index.sizes[i]))
strike.matrix[,i] = c(strikes, rep(NA, max.row - end.index.sizes[i]))
p0 = put.prices[end.index.sizes[i]]
k0 = strikes[end.index.sizes[i]]
p0.vector[i] = p0
k0.vector[i] = k0
put.prices = put.prices[-end.index.sizes[i]]
strikes = strikes[-end.index.sizes[i]]

optim.obj = optim(c(initial.xi, initial.beta), objective.function, c0 = p0, k0 = -k0, call.prices = put.prices, strikes = -1*strikes,
hessian = TRUE, control=list(maxit=10000) )
did.it.converge[i] = optim.obj$convergence
temp.se = sqrt(diag(solve(optim.obj$hessian)))
xi.se[i] = temp.se[1]
beta.se[i] = temp.se[2]

xi.estimates[i] = optim.obj$par[1]
beta.estimates[i] = optim.obj$par[2]

predicted.put.matrix[,i] = p0 * ( (1 + (xi.estimates[i]/beta.estimates[i])*(k0 - strike.matrix[,i]) )^(1 - 1/xi.estimates[i]) )

out = list(xi.estimates = xi.estimates,
           xi.se = xi.se,
           beta.estimates = beta.estimates,
           beta.se = beta.se,
           did.it.converge = did.it.converge,
           dim.sizes = dim.sizes,
           p0.vector = p0.vector,
           k0.vector = k0.vector,
           put.data.only = put.data.only,
           put.matrix = put.matrix,
           predicted.put.matrix = predicted.put.matrix,
           strike.matrix = strike.matrix)

out
\[ \text{xi} = \text{put.object.15}\$\text{xi.estimates}[i] \]
\[ \text{beta} = \text{put.object.15}\$\text{beta.estimates}[i] \]
\[ p0 = \text{put.object.15}\$\text{p0.vector}[i] \]
\[ k0 = \text{put.object.15}\$\text{k0.vector}[i] \]
\[ k = \text{as.numeric(na.omit(put.object.15}\$\text{strike.matrix[,]i\])}) \]
\[ k = k[-\text{length}(k)] \]
\[ \text{put.prices} = \text{as.numeric(na.omit(put.object.15}\$\text{put.matrix[,]i\})}) \]
\[ \text{predicted.values} = \text{as.numeric(na.omit(put.object.15}\$\text{predicted.put.matrix[,]i\})}) \]
\[ \text{predicted.values} = \text{predicted.values}[\text{length}(\text{predicted.values})] \]
\[ \text{model.residuals} = \text{predicted.values} - \text{put.prices} \]
\[ \text{mean.of.residuals} = \text{mean(\text{model.residuals})} \]
\[ \text{relative.residuals} = (\text{put.prices} - \text{predicted.values})/\text{put.prices} \]
\[ \text{mean.adjusted.relative.residuals} = \text{relative.residuals} - \text{mean(\text{relative.residuals})} \]
\[ \text{estimated.sigma[i]} = \sqrt{\text{sum}((\text{model.residuals} - \text{mean.of.residuals})^2)/(\text{length}(\text{model.residuals}) - 2))} \]
\[ \text{estimated.relative.sigma} = \sqrt{\text{sum}((\text{relative.residuals} - \text{mean(\text{relative.residuals})})^2)/(\text{length}(\text{relative.residuals}) - 2))} \]

\text{xi.vector} = \text{numeric}(m)

\text{for (j in 1:m)} { 
\text{resampled.residuals} = \text{sample(\text{mean.adjusted.relative.residuals}, \text{length(\text{mean.adjusted.relative.residuals}), replace = TRUE})} 
\text{y} = p0 \times \{( \text{1 + (\text{xi}/\text{beta})\times(\text{k0} - \text{k}) \times(\text{1 - \text{xi/\text{xi}}}) \times (\text{1 - \text{resampled.residuals}})} \}
\text{optim.obj} = \text{optim(c(\text{xi, beta}), \text{objective.function}, \text{c0} = p0, \text{k0} = -k0, \text{call.prices} = \text{y}, \text{strikes} = -1*\text{k}, \text{hessian} = \text{FALSE}, \text{control=list(maxit=10000)})} 
\text{xi.vector[j]} = \text{optim.obj}\$\text{par}[1] 
}

\text{upper.quantile.xi.estimate[i]} = \text{quantile(\text{xi.vector}, type = 3, prob = 0.975, names = FALSE)}
\text{lower.quantile.xi.estimate[i]} = \text{quantile(\text{xi.vector}, type = 3, prob = 0.025, names = FALSE)}
\text{estimated.bootstrap.se[i]} = \sqrt{\text{sum}((\text{xi.vector} - \text{mean(\text{xi.vector})})^2)/(\text{m - 2})} \}

\text{upper.quantile.xi.estimate.2} = \text{numeric(\text{no.days})}
\text{lower.quantile.xi.estimate.2} = \text{numeric(\text{no.days})}
\text{estimated.bootstrap.se.2} = \text{numeric(\text{no.days})}

\text{m = 1000}

\text{for (i in 1:\text{no.days})} { 
\text{xi.2} = \text{put.object.15}\$\text{xi.estimates}[i] 
\text{beta.2} = \text{put.object.15}\$\text{beta.estimates}[i] 
\text{p0} = \text{put.object.15}\$\text{p0.vector}[i] 
\text{k0} = \text{put.object.15}\$\text{k0.vector}[i] 
\text{k} = \text{as.numeric(na.omit(put.object.15}\$\text{strike.matrix[,]i\])}) 
\text{k} = k[-\text{length}(k)] 
\text{put.prices} = \text{as.numeric(na.omit(put.object.15}\$\text{put.matrix[,]i\})}) 
\text{predicted.values} = \text{as.numeric(na.omit(put.object.15}\$\text{predicted.put.matrix[,]i\})}) 
\text{predicted.values} = \text{predicted.values}[\text{length}(\text{predicted.values})] 
\text{xi.vector.2} = \text{numeric}(\text{m}) 
\text{for (j in 1:\text{m})} { 
\text{case.numbers} = \text{seq(1:\text{length}(\text{k}))} 
\text{resampled.case.numbers} = \text{sample(\text{case.numbers}, \text{length(\text{case.numbers}), replace = TRUE})} 
\text{resampled.k} = k[\text{resampled.case.numbers}] 
\text{resampled.y} = \text{put.prices}[\text{resampled.case.numbers}] 
\text{optim.obj.2} = \text{optim(c(\text{xi.2, beta.2}), \text{objective.function}, \text{c0} = p0, \text{k0} = -k0, \text{call.prices} = \text{resampled.y}, \text{strikes} = -1*\text{resampled.k}, \text{hessian} = \text{FALSE}, \text{control=list(maxit=10000)})} 
\text{xi.vector.2[j]} = \text{optim.obj.2}\$\text{par}[1] 
}

54
upper.quantile.xi.estimate.2[i] = quantile(xi.vector.2, type = 3, prob = 0.975, names = FALSE)
lower.quantile.xi.estimate.2[i] = quantile(xi.vector.2, type = 3, prob = 0.025, names = FALSE)
estimated.bootstrap.se.2[i] = sqrt(sum((xi.vector.2 - mean(xi.vector.2))^2)/(m - 2))

library(tseries)
library(zoo)
vix.sp500.data = read.table("VIX-SP500-data.txt", header=T)
dates = as.Date(vix.sp500.data$DATE)

x11()
par(pch=19, mar = (c(6, 5, 5, 4) + 0.3) )
plot.zoo(zoo(put.object.10$p0.vector, dates) , type="b", main="", ylab="Initial Puts Values (10%)", pch=19, xlab="", cex.axis=1.5, cex.lab=1.5)

x11()
par(pch=19, mar = (c(6, 5, 5, 4) + 0.3) )
plot.zoo(zoo(put.object.15$p0.vector, dates) , type="b", main="", ylab="Initial Puts Values (15%)", pch=19, xlab="", cex.axis=1.5, cex.lab=1.5)

x11()
par(pch=19, mar = (c(6, 5, 5, 4) + 0.3) )
plot.zoo(zoo(put.object.20$p0.vector, dates) , type="b", main="", ylab="Initial Puts Values (20%)", pch=19, xlab="", cex.axis=1.5, cex.lab=1.5)

x11()
par(pch=19, mar = (c(6, 5, 5, 4) + 0.3) )
plot.zoo(zoo(put.object.10$k0.vector, dates) , type="b", main="", ylab="Initial Strike Values (10%)", pch=19, xlab="", cex.axis=1.5, cex.lab=1.5)

x11()
par(pch=19, mar = (c(6, 5, 5, 4) + 0.3) )
plot.zoo(zoo(put.object.15$k0.vector, dates) , type="b", main="", ylab="Initial Strike Values (15%)", pch=19, xlab="", cex.axis=1.5, cex.lab=1.5)

x11()
par(pch=19, mar = (c(6, 5, 5, 4) + 0.3) )
plot.zoo(zoo(put.object.20$k0.vector, dates) , type="b", main="", ylab="Initial Strike Values (20%)", pch=19, xlab="", cex.axis=1.5, cex.lab=1.5)

x11()
par(pch=19, mar = (c(6, 5, 5, 4) + 0.3) )
plot.zoo(zoo(ceiling( 0.10 * put.object.10$dim.sizes), dates) , type="b", main="", ylab="Number of Puts (10%)", pch=19, xlab="", cex.axis=1.5, cex.lab=1.5)

x11()
par(pch=19, mar = (c(6, 5, 5, 4) + 0.3) )
plot.zoo(zoo(ceiling( 0.15 * put.object.15$dim.sizes), dates) , type="b", main="", ylab="Number of Puts (15%)", pch=19, xlab="", cex.axis=1.5, cex.lab=1.5)

x11()
par(pch=19, mar = (c(6, 5, 5, 4) + 0.3) )
plot.zoo(zoo(ceiling( 0.20 * put.object.20$dim.sizes), dates) , type="b", main="", ylab="Number of Puts (20%)", pch=19, xlab="", cex.axis=1.5, cex.lab=1.5)

xi.values.together = cbind(put.object.10$xi.estimates, put.object.15$xi.estimates, put.object.20$xi.estimates)
min.plot.value = min(xi.values.together)*1.05
max.plot.value = max(xi.values.together)*1.05
zoo.tmp = zoo(xi.values.together, dates)
plot.zoo(zoo.tmp, plot.type = "single", ylab="Xi", xlab="", type="b", pch=c(16,18,17), cex.axis=1.5, cex.lab=1.5,
lty=c(1,1,1), lwd = c(2,2,2), col=c("red","blue","green"), ylim=c(min.plot.value, max.plot.value))
legend("bottomleft", legend=c("10%","15%","20%"), col=c("red","blue","green"), lty=c(1,1,1), lwd = c(2,2,2), bty="n", cex=1.5, pch=c(16,18,17))
### S&P 500 and xi at 15% "sp500-xi-15-together"

```r
par(mar = (c(6, 5, 5, 4) + 0.3))
zoo.tmp.2 = zoo(cbind(vix.sp500.data$SP500, put.object.1S$xi.estimates), dates)
plot.zoo(zoo.tmp.2, type="b", main="", ylab=c("SP500", "Xi (15%)"), pch=19, xlab="", cex.axis=1.5, cex.lab=1.5, lwd=c(2,2), col=c("black","blue"))
```

# One Day Analysis September 8, 2008, 54th day

```
min.open.interest = 0
min.bid.price = 0
put.data.only = subset(op.data, cp_flag == "P")
put.data.only = transform(put.data.only, strike_price = strike_price/1000)
put.data.only = subset(put.data.only, open_interest > min.open.interest)
put.data.only = subset(put.data.only, best_bid > min.bid.price)

temp = subset(put.data.only, date == "20080908" , select = c(strike_price, option_price ))
put.data.20080908.day = unique(temp[order(temp$strike_price),])
len.20080908.day = dim(put.data.20080908.day)[1]
end.index.size.20080908.day = ceiling(len.20080908.day * lower.q)
lower.put.prices = put.data.20080908.day$option_price[1:end.index.size.20080908.day]
lower.put.strike.prices = put.data.20080908.day$strike_price[1:end.index.size.20080908.day]
```

### Plot all of put values , in file 'oneday-corrected'

```
x11(height = 7, width = 7)
par(mar = c(5, 5, 4, 3) + 0.15, cex.lab = 1.4, cex.axis = 1.4, pch=20)  # default if c(5, 4, 4, 2) + 0.1
plot(option_price ~ strike_price, data = put.data.20080908.day , xlab="Strike Values", ylab="Put Values")
abline(v = rev(lower.put.strike.prices)[1], lty = 2, lwd = 0.5)
```

### Plot all of put values for the lower 15%

```
x11(height = 7, width = 7)
par(mar = c(5, 5, 4, 3) + 0.15, cex.lab = 1.4, cex.axis = 1.4, pch=19)
plot(lower.put.prices ~ lower.put.strike.prices, xlab="Strike Values", ylab="Put Values")
```

```r
xi.20080908.day = put.object.15$xi.estimates[54]
beta.20080908.day = put.object.15$beta.estimates[54]
x = seq(min(lower.put.strike.prices), max(lower.put.strike.prices), length.out = 250)
y = max(lower.put.prices) * ((xi.20080908.day/beta.20080908.day)*(max(lower.put.strike.prices) - x) + 1)^(1 - 1/xi.20080908.day)
lines(x,y)
```

### Plot of xi and xi with ci

```
x11(height = 7, width = 7)
par(mar = c(5, 5, 4, 3) + 0.15)  # default if c(5, 4, 4, 2) + 0.1
plot.zoo(xi.estimates.zoo.15, type="l", xlab="Day Shape", main ="", lty = c(2,1,2), col=c("black","red","black"), xlab="", cex.axis=1.5, cex.lab=1.5)
```

save.image("sp500-put-analysis-october-2011.RData")
```
H Sample Analysis

The purpose of this appendix is to present the process of recovering the tail shape parameter from a set of given OTM put options in a step by step manner using software. The OTM put options are generated from the BSM formula in the equation (25). The BSM parameter values are the same as the ones used in the simulation analysis of Section (6).

To reproduce the results in this section, just copy and paste the code onto the R GUI. R’s syntax is similar to Matlab and other popular software so the code presented here can be used as pseudo-code for other languages. The code is presented in a different font and separated with extra space to distinguish it from the body of the text.

We first define two functions of interest: a function to generate prices from the BSM model, and a function to evaluate OTM put options based on the GP model. The second function will be passed into R’s optimization function to obtain estimates of (or recover) the tail shape and scale parameters by minimizing the relative error. This function is written for calls but can be used for puts as demonstrated later.

```
###############################################################
#### Black-Scholes-Merton Option Pricing Formula with dividend yield as well
###############################################################
#### Inputs:
#### r = risk free rate, te = time to expiration, S0 = current price
#### k = strike, sigma = volatility, y = dividend yield
#### (main) Outputs
#### call = call prices, put = put prices
####
#### bsm.option.price = function(r, te, S0, k, sigma, y)
#### {
####  d1 = (log(S0/k) + (r - y + (sigma^2)/2) * te) / (sigma * sqrt(te))
####  d2 = d1 - sigma * sqrt(te)
####  call.option.price = S0 * exp(-y*te) * pnorm(d1) - k * exp(-r*te) * pnorm(d2)
####  put.option.price = k * exp(-r*te) * pnorm(-d2) - S0 * exp(-y*te) * pnorm(-d1)
####  out = list(d1 = d1, d2 = d2, call = call.option.price, put = put.option.price)
####  out
#### }

###############################################################
#### This is the objective function for minimizing the relative error.
####
#### objective.function = function(theta, c0, k0, call.prices, strikes)
#### {
####  x1 = theta[1]
####  beta = theta[2]
####  sample.size = length(call.prices)
####  check.terms = 1 + (x1/beta)*(strikes - k0)
####  presence.of.neg.indicator = sum(check.terms <= 0)
####  if ( presence.of.neg.indicator > 0 ) (out = 10^7) else { y = call.prices
####  x = (c0) * ( (check.terms)^(1 - 1/x1) )
####  out = sum(abs(1 - x/y))
####  out
#### }
```

57
We generate 11 OTM put options based on the BSM model. The strikes range from 900 to 850 by the decrements of 5.

\[
\text{strike.range} = \text{seq}(\text{from} = 900, \text{to} = 850, \text{by} = -5)
\]

\[
\text{bsm.put.values} = \text{bsm.option.price}(r = 0.03, \text{ts} = 50/265, S0 = 1000, \\
\quad k = \text{strike.range}, \sigma = 0.20, y = 0.02)\pl{put}
\]

The prices of the OTM puts range from $4.28 to $0.90. The greater than sign in the code below is R’s command prompt and should not be copied onto the R GUI. Copy and paste the command “round(bsm.put.values, 2)”. The second line is just the output.

\[
> \text{round(bsm.put.values, 2)} \\
[1] 4.28 3.73 3.24 2.80 2.41 2.07 1.77 1.50 1.27 1.07 0.90
\]

Next, we recover the tail shape and scale parameters by minimizing the distance between the BSM OTM puts and the GP based model. The initial values were roughly equal to the averages of the estimates from the analysis of S&P 500 put options.

\[
p0.\text{star} = \text{bsm.put.values}[1] \\
k0 = 900
\]

\[
\text{initial.values} = \text{c}(0, 100)
\]

\[
\text{optim.obj} = \text{optim}(\text{initial.values, objective.function, } c0 = p0.\text{star}, k0 = -k0, \\
\quad \text{call.prices} = \text{bsm.put.values, strikes} = -1*\text{strike.range}, \\
\quad \text{hessian} = \text{F, control=}{\text{list}}(\text{maxit}=10000) )
\]

We can look at the parameter estimates as follows. The estimate for the tail shape parameter is \( \hat{\xi} = -0.2163655 \), and the scale parameter \( \hat{\beta}(K_0) = 44.5561244 \). The output also shows some additional information about the optimization performance.

\[
> \text{optim.obj}$par \\
\{1\} -0.2163655 44.5561244
\]

\[
> \text{optim.obj}$value \\
\{1\} 0.005902488
\]

\[
> \text{optim.obj}$counts \\
\{1\} 117 NA
\]

\[
> \text{optim.obj}$convergence \\
\{1\} 0
\]

\[
> \text{optim.obj}$message \\
\text{NULL}
\]

Next, we obtain the GPD based prices for the strikes 900, 895, \ldots, 855, 850 and compare them.
Figure 9: Plot of the BSM OTM put values versus their strikes (in dark points) and the line fitted by the GPD based pricing model.

to the BSM prices side by side. Note that the GP based values are very close to the BSM prices.

Finally, we plot the BSM put prices versus the strikes and add the pricing line implied by the GPD model. Note the excellent fit in figure 9.
I Pricing New OTM Options

Suppose on September 8, 2008, a trader needs to price an OTM put option on the S&P 500 Index with a strike of 925 and expiring on December 20, 2008. No such option exists. Typically, the trader’s first step would be to interpolate the implied volatility for an option of strike 925 from the nearby traded options. Then this implied volatility, along with the current index value, the expiration time, the current dividend yield, and the risk free rate are plugged into the BSM pricing formulas to obtain the price.

Using the recovered tail shape and scale parameters from the traded options listed on table (2), we can price this new OTM put option with a strike of 925 as follows:

\[
\text{put price using 15\%} = (4.60) \left[ \frac{0.0701}{77.0734} (1000 - 925) + 1 \right]^{1-1/(0.0701)} = \$1.92.
\]

This price simply comes from the curve fitted to the 9 OTM put options. Had we chosen the initial put to be $10.45 with a strike 1075 (not shown in table (2)) so that 20\% of the lowest puts were selected, we would have estimated \( \hat{\xi}' = 0.0407, \hat{\beta}' = 82.9741 \). The price in this case is:

\[
\text{put price using 20\%} = (10.45) \left[ \frac{0.0407}{82.9741} (1075 - 925) + 1 \right]^{1-1/(0.0407)} = \$1.96.
\]
A Brief Literature Review of Risk Neutral Density Estimation

The methods for the estimation of the RND can be divided into three groups: parametric, semi-parametric, and non-parametric.


Many of the semi-parametric methods are based on the Edgeworth expansion of the unknown RND around the lognormal density which was first proposed by Jarrow and Rudd (1982). Another group of semi-parametric methods, which are based on the Hermite polynomial expansion of the normal density, is due to Madan and Milne (1994). For the application of these semi-parametric based methods see Abken et al. (1996), Giacomini et al. (2009), Rompolis and Tzavalis (2008), Flamouris and Giamouridis (2002), and Jondeau and Rockinger (2001).

The use mixture of distribution (generally lognormal) is a popular parametric method. Bahra (1997) gives applications of a mixture of two-lognormal to various options. Melick and Thomas (1997) propose a method for recovering the density from the American option prices by incorporating the theoretical bounds on the American options. Parametric methods using various distributions or stochastic processes for the underlying include: Liu et al. (2007) with the generalized beta distribution, Abadir and Rockinger (2003) with a mixture of hypergeometric functions that can embed
many known densities, and Tunaru and Albota (2005) with the generalized gamma distribution.

Comparison of the performance of different estimation methods have been made in a number of works. Using options on foreign exchange data, Jondeau and Rockinger (2000) test semi-parametric approaches based on Hermite and Edgeworth expansions, single and mixture lognormals, and methods that assume underlying jump diffusion and stochastic volatility models. They recommend using the mixture of lognormals model for the short-run options, and the jump-diffusion model for the long-run options. Coutant et al. (2001) compare methods based on Hermite expansion, maximum entropy, mixture of lognormals, and a single lognormal and conclude that all methods do better than a single lognormal; they favor the Hermite expansion method due to its numerical speed, stability, and accuracy. Bliss and Panigirtzoglou (2002) compare their spline-based method with a parametric method of mixture of two lognormals and conclude that theirs outperforms the double log-normal. In a Monte Carlo experiment, Bondarenko (2003) shows that his non-parametric method beats the second best method of mixture of three lognormals among a number of competing methods. Mandler (2003), pages 109–110, concludes that the computational convergence rates of the mixture of distributions and the maximum entropy are slow relative to the expansion methods.

A number of studies have focused on the interpretation of the RND. Some examples include Beber and Brandt (2006) with a focus on the effect of the scheduled macroeconomic announcements on the participants in the U.S. Treasury market, Shiratsuka (2002) and Nakamura and Shiratsuka (1999) with a focus on the predictive power of the RND estimated from the Japanese derivative markets, Mandler (2002) with a focus on the influence of ECB meetings on market expectations based on the LIFFE-Euribor futures options, Levin et al. (1998) with a focus on the information content of the Canadian Dollar futures options, and Campa et al. (1997) with a focus on the determination of exchange rate expectations.