Gambling in Dynamic Contests

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Abstract This paper presents a strategic model of risk-taking behavior in the framework of a continuous time contest. Formally, we analyze a dynamic game in which each player decides when to stop a privately observed Brownian Motion with drift. Only the player who stops his process at the highest value wins a prize. We derive a closed-form solution for the unique Nash equilibrium outcome in mixed strategies and we establish that the expected value of the stopped stochastic processes is non-monotone in the drift. In particular, the highest losses in terms of expected value occur if the drift is only moderately negative. Thus, relative performance payments, while suitable to provide the right incentives in good times, induce socially undesirable gambling activities if times are moderately bad. Possible applications of the model include contests for status, job promotion contests, competition between mutual funds, and relative payment schemes of CEOs.

Keywords: Discontinuous Games, Dynamic Contests, Relative Performance Pay, Risk-Taking Behavior

JEL code: C72, C73, D81

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1 Introduction

In many situations economic agents have the opportunity to invest other agents' money in risky projects. Examples include top managers or CEOs of firms who face risk-taking decisions for the money of shareholders, other lenders or creditors. Mutual fund companies receiving money from fund investors are another example.

Over the last decades, there has been a large amount of literature on finding optimal payment schemes in general principal-agent frameworks with imperfect observability. In their seminal paper, Lazear and Rosen (1981) show in a simple static model that tournaments (e.g. for job promotion or monetary premia) may induce efficient incentives. More recently, the Relative Performance Evaluation (RPE) Hypothesis claims that in industries with common shocks CEOs should be paid according to their performance relative to the industry. The underlying idea is to shift uncertainty in the market from risk-averse CEOs to risk-neutral firms.

In the empirical literature, there is some evidence for indirect rewards for relative performance. For instance, CEOs tend to receive more job offers if they have achieved better results than other CEOs in the same industry (Baker et al., 1988 and Brown et al., 1996). Furthermore, the inflow of new investments in mutual funds is strongly correlated with the funds past performances (Chevalier and Ellison, 1997, 1999 and Taylor, 2003).

However, despite the aforementioned theoretical arguments in favor of contests, there is very limited evidence on direct relative incentive schemes for CEOs. In 1992, Janakiraman et al. conclude from their empirical observations that “the agency model used to develop the RPE predictions is not descriptively valid for CEOs” (p.67). Aggarwal and Samwick (1999, p.104) state that “relative performance evaluation considerations are not incorporated into executive compensation contracts”. According to the survey article by Murphy (1999, p.2539), “the paucity of RPE in options and other components of executive compensation remains a puzzle worth understanding”.

As a contribution to solving this puzzle, this paper provides a theoretical argument against relative performance payments based on risk-taking con-
siderations. For this purpose, we intend to capture the most crucial common features of the examples. Most importantly, agents are faced with tournaments in stochastic environments which usually last at least one year (rather than one-shot interactions). Differing from other principal-agent models or R&D contest, we think that in the applications we aim to model effort costs are mostly fixed or small compared to potential risk effects, because the costly information acquisition is carried out by other agents (advisors).

In order to take the previous considerations into account, we assume that (i) effort costs are negligible or fixed, (ii) participants do not bear monetary losses from a lower contest success function (only relative success counts) and (iii) compete in a stochastic environment over a longer period of time incorporating past information in their future decisions. More precisely, we propose a continuous time contest model, in which agents dynamically decide on stopping a stochastic process (Brownian Motion with drift) under the restriction that they are forced to stop in case of bankruptcy (if the process reaches zero). Not stopping the process can be interpreted as continuing to invest in a risky project.

In accordance with most of the contest literature, stopping decisions of other players are assumed to be unobservable. At the end of the game, the player who stopped his process at the highest value is awarded a prize.

In the main part of this paper, we derive the unique Nash equilibrium distribution induced by the stopping strategies of the players. For any value of the drift, there is no equilibrium in which agents stop the process immediately. Intuitively, even if agents lose money in expectation (negative drift), each player can reach a value slightly above the starting value with high probability if he plays until reaching this value or going bankrupt. Surprisingly, we find expected equilibrium losses to be maximal in moderately bad times, i.e. if expected losses per time are relatively small. Consequently, a decrease of expected losses per time might entail an increase of equilibrium losses.

An important policy implication is that relative performance payments may induce fund managers and CEOs to hold assets or start projects with a negative expected value. The expected waste of money is highest if times are bad, but not too bad. Moreover, the equilibrium structure shows a remark-
able discontinuity if drift and variance approach zero. This discontinuity illustrates a general lack of robustness in relative performance based mechanisms. Accordingly, each principal who is not perfectly informed about drift and variance might incur huge losses through these mechanisms.

Summarizing, this paper provides a possible explanation why relative performance payments are not commonplace in CEO compensation schemes and related problems. Furthermore, it predicts that mutual funds invest excessively in unprofitable risky projects in times of moderately decreasing markets. Note that the whole reasoning hinges on the dynamic structure and cannot be obtained in simple one-shot two-player frameworks à la Lazear and Rosen (1981).

From a technical point of view, the contribution of this paper is the introduction of a solution method for continuous time contest models. In a first step, we derive the unique candidate for Nash equilibrium distributions using indifference conditions for all players. A solution technique new to economic research is introduced in the second step, where we prove that these distributions are indeed feasible, i.e. it is possible to induce these distributions by a finite time stopping strategy. In the proof, we rely on a necessary and sufficient condition introduced in a recent paper in probability theory by Grandits and Falkner (2000). Due to its generality and analytical tractability, our approach could be useful to solve other related models.

2 Related Literature

Inefficiency results similar to the present paper are obtained in the literature on “gambling for resurrection” (e.g. Downs and Rocke, 1994). The basic idea is that a manager only cares if his firm is able to survive which leads him to risk money in gambles with negative expected value to save the firm from bankruptcy. However, the reasoning for participating in “unfair” gambles in the present approach is different, as players want to “veil” their final outcome to be unpredictable for the opponent.

Hvide (2002) alludes to the problem of excessive risk-taking in a one-period contest model. In his model, agents costlessly choose their variance,
but bear costs to increase their expected return. In equilibrium all agents choose the highest possible variance and very low effort levels (in the basic model an infinite variance and no effort). Due to its dynamic structure, the present model allows for more realistic predictions of risk-taking behaviour, i.e. agents do not maximize the variance of their final outcome.

A related model of a contest is Taylor (1995). In his discrete time model, agents decide in every period on taking a costly draw from a common distribution. The highest draw yields the victory in the tournament. Taylor finds that there is a unique symmetric equilibrium outcome in deterministic cutoff strategies. This finding is in sharp contrast to the present paper, where the equilibrium strategies are fully mixed. The differing predictions result from the additive way of modelling contest success in this paper compared to a running maximum process in Taylor (1995).

In the literature on races, the most closely related model is Moscarini and Smith (2007). Building on an earlier model of Harris and Vickers (1987), they analyze a two-person continuous-time race with costly effort choice. They find that the effort is increasing in the lead of a player up to some threshold after which the laggard gets discouraged. Similarly, Gul and Pesendorfer (2009) discuss a political economy model where two agents (politicians) control a common stochastic process. Again, the laggard stops (his political campaign) if the lead of the other player exceeds a deterministic threshold. The crucial difference to the present approach is that players can observe the realizations of each other’s contest success function over time, which leads to resignation if the lead of the other player is too high.

The rest of the paper is organized as follows: Section 3 formally introduces the model. The technical analysis is provided in Section 4. Comparative statics results and their implications as well as related literature are discussed in Section 5. Section 6 concludes.
3 The Model

3.1 The Setting

We consider a model in which \( n \) agents \( i \in \{1, 2, \ldots, n\} = N \) face a continuous time stopping problem. At each point in time \( t \in \mathbb{R}_+ \), agent \( i \) privately observes the realization of a stochastic process \( X^i = (X^i_t)_{t \in \mathbb{R}_+} \) which is given by

\[
X^i_t = x_0 + \mu t + \sigma B^i_t.
\]

Here \( B^i_t \) is a standardized Brownian motion, i.e. \( B^i_0 = 0 \), \( B^i_t \) is continuous almost surely and \( B^i_t \) has independent normally distributed increments \( B^i_{t+\Delta} - B^i_t \sim \mathcal{N}(0, \Delta) \). We assume that there is no correlation between the \( B^i_t \)'s. \( \sigma t \) is the standard deviation of the random variable \( X^i_t \). The constant \( x_0 > 0 \) is the initial value of all processes (heterogeneous initial values are discussed in Section 4.3) and \( \mu \in \mathbb{R} \) is the expected change of the process \( X^i_t \) per time, i.e.

\[
\mathbb{E}(X^i_{t+\Delta} - X^i_t) = \mu \Delta.
\]

The process \( X^i_t \) can be interpreted as the contest success function of player \( i \) at time \( t \). It is determined as the sum of his initial endowment \( x_0 \), a deterministic component \( \mu t \) and a normally distributed noise term \( \sigma B^i_t \).

3.2 Strategies

Player \( i \) can make an irreversible stopping decision at any time \( t \geq 0 \) with the restriction that whenever \( X^i_t = 0 \) (bankruptcy) he is forced to stop. Formally

\[
\tau^i \leq \inf\{t \in \mathbb{R}_+ : X^i_t = 0\},
\]

where \( \tau^i \) denotes the stopping time of player \( i \). Agent \( i \) can condition his stopping decision only on past realizations of his stochastic process. Mathematically, the stopping time \( \tau^i \) must be such that for every \( t \) the event that the agent has stopped is measurable, i.e \( \{\tau < t\} \in \mathcal{F}^i_t \), where \( \mathcal{F}^i_t = \sigma(\{X^i_s : s < t\}) \) is the sigma algebra induced by possible observations.
of the process $X^i$ before time $t$. We restrict agents’ strategy spaces to finite time stopping strategies, i.e. require $\tau^i < \infty$ almost surely.

A randomized stopping time is a progressively measurable function $\tau^i(\cdot)$ such that for every $r^i \in [0, 1]$, $\tau^i(r^i)$ is a stopping time. Intuitively, this specification allows for mixed strategies in which agents draw a random number $r^i$ from the uniform distribution on $[0, 1]$ before the game and then play the stopping strategy $\tau^i(r^i)$.

Note that this model is equivalent to a situation where the agents stopping decisions are reversible and the stopped processes are constant. The crucial assumption we impose is that agents do not observe anything but their own process.

3.3 Payoffs

There is one positive prize (which is set to one w.l.o.g.) that is awarded to agent $i$ if his stopped process has a higher value than the stopped processes of all other players. In case of a tie, the prize is randomly assigned among all players with the highest value. Formally, the payoff function $\pi^i$ is defined as

$$\pi^i = \frac{1}{k} \mathbb{1}_{\{X^i_{\tau^i} = \max_{j \in N} X^j_{\tau^j}\}}$$

where $k = \left| \{i \in N : X^i_{\tau^i} = \max_{j \in N} X^j_{\tau^j}\} \right|$. All agents maximize their expected payoff, i.e. the probability to win the contest. This optimization is independent of their attitude towards risk. Note that the game is essentially a zero sum game because $\sum_{i=1}^n \pi^i = 1$.

3.4 Condition on the Parameters

To ensure equilibrium existence in finite time stopping strategies, we henceforth impose a technical condition which puts a positive upper bound on $\mu$ (for a discussion see Section 4.2):

**Assumption 1.** $\mu < \ln(1 + \frac{1}{n-1}) \frac{\sigma^2}{2\sigma^2}$. 

4 Equilibrium Analysis

In this section, we first derive the unique candidate for equilibrium distribution functions. However, as the game in consideration has discontinuous payoffs and infinite strategy space, equilibrium existence needs to be shown separately. To do so, we prove the existence of stopping strategies inducing the unique equilibrium distributions. The section is closed with an extension to asymmetric starting values.

4.1 The Equilibrium Distribution

Every strategy played by agent $i$ induces a (potentially non-smooth) cumulative distribution function (cdf) $F^i : \mathbb{R}_+ \rightarrow [0, 1]$ of his stopped process, where $F^i(x) = \mathbb{P}(X^i_{\tau_i} \leq x)$.

The probability that at least two agents stop at the same highest value can only be positive if there exist at least two agents whose final distributions have a mass point above zero or all agents distributions have a mass point at zero. However, the next lemma, the proof of which is relegated to the appendix, states that this is not the case.

**Lemma 1 (No Mass Points).** In equilibrium for every $x > 0$ no agent $i \in N$ has a mass point at $x$, i.e. $\mathbb{P}(X^i_{\tau_i} = x) = 0$. At least one agent has no mass point at zero.

Thus, the tie-breaking rule can henceforth be ignored, because the probability that more than one player ends up at the highest stopping value is zero. Using the fact that there are no mass points away from zero, we can express the utility of agent $i \in N$ for every $x > 0$ in terms of the cdf of the other players. For this purpose, let $u^i(x) : \mathbb{R}_+ \rightarrow [0, 1]$ denote the expected payoff of agent $i$ of stopping at value $X^i_{\tau_i} = x$, given the strategies of the other players.

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1As common in the economic literature we do not consider the mathematical problem of an accumulation of mass points (Cantor Construction) and assume that either there is only a finite number of mass points or they have no accumulation point.
\[
u^i(x) = P(x \geq \max_j X^j_{\tau}) + \frac{1}{k} \sum_{j \neq i} P(x = \max_j X^j_{\tau}) = 0
\]

\begin{equation}
= \prod_{j \neq i} P(X^j_{\tau} \leq x) = \prod_{j \neq i} F^j(x)
\end{equation}

One can interpret \(u^i(\cdot)\) as the utility function induced on agent \(i\) by the other agents equilibrium distributions. Thus, taking as given the equilibrium distributions of the other players, the problem of agent \(i\) reduces to a single-person stopping problem of maximizing \(E(u^i(X^i_\tau))\).

In the following paragraphs, this consideration is used to derive the equilibrium. First, we denote the right endpoint of the support of the distribution of player \(i\) by \(\bar{x}^i = \sup\{x : F^i(x) < 1\}\) and the left endpoint by \(\underline{x}^i = \inf\{x : F^i(x) > 0\}\). Existence of the endpoints follows from the fact that agents can only use strategies that stop almost surely in finite time. The following results establish necessary equilibrium conditions on \(u^i\) and the distribution functions (see appendix for the proofs).

**Lemma 2** (Strict Monotonicity). *In equilibrium the utility \(u^i\) of every agent \(i \in N\) is strictly increasing on the interval \([\underline{x}^i, \bar{x}^i]\).*

**Lemma 3** (Indifference). *For each player \(i\), the utility \(u^i(X^i_\tau)\) is a local martingale on the interior of the support of his distribution, i.e \(X^i_\tau \in (\underline{x}^i, \bar{x}^i) \Rightarrow E(d(u^i(X^i_\tau))) = 0\).*

**Lemma 4** (Symmetry of the Support). *The support of the cdf of all players is identical and given by \([0, \bar{x}]\).*

Furthermore, because all players share the same utility function, Lemma 3 and 4 directly imply the following corollary:

**Corollary 1.** *The unique equilibrium distributions are atomless and symmetric.*

Noticing that the utility \(u^i\) does not depend on time (\(\frac{\partial u^i}{\partial t} = 0\)), the expected change in utility per marginal unit of time is calculated using Itô’s lemma (Revuz and Yor, 2005, p.147) as
$E(\text{d}u^i(x)) = \mathbb{E}\left( \mu \frac{\partial u^i}{\partial x}(x) + \sigma^2 \frac{\partial^2 u^i}{(\partial x)^2}(x) \right) dt + \frac{\partial u^i}{\partial x}(x) \sigma dB_t$

$$= \mathbb{E}\left( \mu \frac{\partial u^i}{\partial x}(x) + \frac{\sigma^2}{2} \frac{\partial^2 u^i}{(\partial x)^2}(x) \right) dt.$$ 

To satisfy the condition of Lemma 3, we set this equation equal to zero and obtain the following ordinary differential equation

$$0 = \mu \frac{\partial u^i}{\partial x}(x) + \frac{\sigma^2}{2} \frac{\partial^2 u^i}{(\partial x)^2}(x).$$

It is easily seen that for $\mu \neq 0$ all solutions to this equation are of the form $u^i(x) = \alpha + \beta \exp\left(\frac{-2\mu x}{\sigma^2}\right)$ for all constants $\alpha, \beta \in \mathbb{R}$. The constants $\alpha$ and $\beta$ are fixed by the additional constraints we derive for $u^i$. In particular, all players win with probability $\frac{1}{n}$ when stopping at their initial value by indifference in a symmetric equilibrium (Lemma 3 and Corollary 1). Furthermore, the value of the cdf is zero at zero by symmetry (Corollary 1) and the fact that not all players can put a mass point on zero (Lemma 1). Thus, we get

$$0 = u^i(0) = \alpha + \beta$$

$$\frac{1}{n} = u^i(x_0) = \alpha + \beta \exp\left(\frac{-2\mu x_0}{\sigma^2}\right).$$

This system of equations uniquely determines $\alpha$ and $\beta$ and the resulting utility $u^i$ is given by

$$u^i(x) = \min \left\{ 1, \frac{1}{n} \exp\left(\frac{-2\mu x}{\sigma^2}\right) - 1 \right\}.$$ 

It remains to construct the equilibrium distributions which induce utilities of the above form. From equation (1) and the symmetry property of the equilibrium (Corollary 1), the final distributions can be easily obtained by taking the $(n-1)$th root of $u^i(x)$,

$$u^i(x) = \prod_{j \neq i} F_j(x) = F(x)^{n-1} \Rightarrow F(x) = \left(\frac{1}{n} \exp\left(\frac{-2\mu x}{\sigma^2}\right) - 1 \right)^{\frac{1}{n-1}} u^i(x).$$
This allows us to characterize the unique candidate for an equilibrium distribution (for an illustration see Figure 1):

**Proposition 1.** Assume $\mu \neq 0$. If a strategy profile is a Nash equilibrium, each player’s strategy induces the cumulative distribution function $F$ given by

$$F(x) = \min \left\{ 1, \, n^{-\frac{1}{2}} \sqrt{\frac{1}{n} \exp\left(-\frac{2\mu x}{\sigma^2}\right) - 1} \right\}.$$  

**Proof.** In the preceding paragraph we have proven that any equilibrium strategy must be symmetric and induce the distribution $F$. To complete the proof, we need to show that there exists no deviation that gives a player a probability of winning greater than $\frac{1}{n}$. Recall that $F$ is constructed in a way which ensures that $u'(X_{i,t})$ is a supermartingale. By Doobs optional stopping theorem (Revuz and Yor, 2005, p.70) the stopped process $u'(X_{i,\tau})$ is also a supermartingale and it follows that $\mathbb{E}(u'(X_{i,\tau})) \leq \mathbb{E}(u'(x_{i,0})) = \frac{1}{n}$. \hfill $\square$

Hence, we have characterized the unique candidate for an equilibrium distribution for each player if $\mu \neq 0$ (for an illustration see Figure 1). To complete the analysis, it remains to consider the special case that the process $X_i$ is a martingale, i.e. $\mu = 0$. Using the same sequence of arguments as for $\mu \neq 0$, it follows that the unique equilibrium distribution of each player is of the following form

$$F^i(x) = \min \left\{ 1, \, n^{-\frac{1}{2}} \sqrt{\frac{x}{nx_0}} \right\}.$$

### 4.2 Equilibrium Strategies

So far, we have been silent about the existence of a finite time stopping strategy $\tau$ inducing the equilibrium distribution $F$. Obviously, a distribution can only be feasible in finite time stopping strategies if its right endpoint is finite. Recall that

$$1 = F(x) = \sqrt[n^{-1}]{\frac{1}{n} \exp\left(-\frac{2\mu x}{\sigma^2}\right) - 1}.$$  

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Figure 1: An example ($\mu = -0.1$, $x_0 = 100$, $\sigma = 1$) for the equilibrium cdf’s depending on the number of players $n$.

Rewriting this equation, we obtain an expression for right endpoint $\bar{x}$ as

$$\bar{x} = \frac{\sigma^2}{-2\mu} \log(n\exp\left(-\frac{2\mu x_0}{\sigma^2}\right) - 1) + 1).$$

Consequently, the right endpoint is finite if and only if the technical condition on the parameters $\mu < -\ln(1 - \frac{1}{n})\frac{\sigma^2}{2x_0}$ introduced in Section 3.4 is met. As $F$ is the unique candidate for an equilibrium distribution no equilibrium in finite time stopping strategies exists otherwise. Intuitively, if the drift term get too large (which seems very unlikely in the applications), players would prefer to continue playing forever and reaching infinity with a probability above $\frac{1}{n}$.

Given the above condition holds, we explicitly construct mixed equilibrium strategies inducing the distribution $F$ in the two-player case to convey the main intuition for an equilibrium construction. We then show feasibility for the $n$-player case relying on a recent mathematical contribution by Grandits and Falkner (2000).
Intuitively, the equilibrium construction for two players uses a mixture of
deterministic threshold strategies to induce the final distribution. To formalize
this intuition, we first introduce the martingale transformation $\phi : \mathbb{R}_+ \to \mathbb{R}_+$
defined as
$$
\phi(x) = \exp\left(\frac{-2\mu x}{\sigma^2}\right) - \exp\left(\frac{-2\mu x_0}{\sigma^2}\right) - 1.
$$
Applying Itô’s lemma, it is easily seen that the process $(\phi(X^i_t))_{t \in \mathbb{R}_+}$ is a
martingale.

**Proposition 2** (Equilibrium Strategy for Two Players). If agent $i$ randomly
chooses a number $\alpha \in [0, 1]$ from a uniform distribution and stops if
$$
\tau^i = \inf\{t : |\phi(X^i_t) - 1| \geq \alpha\},
$$
then the cumulative distribution function induced by this strategy equals $F$,
i.e. $P(X^i_{\tau^i} \geq x) = F(x)$.

*Proof.* By the martingale property of $(\phi(X^i_t))_{t \in \mathbb{R}_+}$, we get
$$
P(\phi(X^i_{\tau^i}) = 1 - \alpha) = P(\phi(X^i_{\tau^i}) = 1 + \alpha) = \frac{1}{2}.
$$
As $\alpha$ is uniformly distributed on $[0, 1]$ and agent $i$ stops iff $\phi(X^i_t) = 1 \pm \alpha$
the random variable $\phi(X^i_{\tau^i})$ is uniformly distributed on $[0, 2]$. It follows that
$$
P(X^i_{\tau^i} \leq x) = P(\phi(X^i_{\tau^i}) \leq \phi(x)) = \frac{\phi(x)}{2} = F(x).
$$

For the general case $n \geq 2$, the feasibility proof relies on more abstract
results. The problem of deciding whether a certain distribution is feasible
is known in the probability theory literature as the Skorokhod embedding
problem. If a probability distribution $F$ is feasible, it is said that it can be
embedded in a Brownian motion with drift. The problem was first defined
and solved for the martingale case in Skorokhod (1961, 1965). In a recent
contribution, Grandits and Falkner (2000) derive a necessary and sufficient
condition for the problem with drift. The following proposition verifies that this condition is met:

**Proposition 3** (Feasibility of the Equilibrium Distribution). *There exists a stopping strategy which induces the distribution\(^{2}\)*

\[
F(x) = \min \left\{ 1, \sqrt[n-1]{\frac{1}{n} \exp\left(\frac{-2\mu x}{\sigma^2}\right) - 1} \right\}.
\]

Proof. To verify the condition in Grandits and Falkner (2000), recall that the stochastic process \((\phi(X^t_i))_{t \in \mathbb{R}_+}\) is a martingale. Consequently for \(f\) to be embeddable it is necessary that the expected value of \(\phi(x)\) under the distribution \(f\) is \(\phi(x_0) = 1\). Recall that \(F(x) = \sqrt[n]{\frac{1}{n} \phi(x)}\).

\[
\mathbb{E}_f(\phi(x)) = \int_{\mathbb{R}} f(x) \phi(x) dx = \int_{0}^{\infty} \left( \frac{n-1}{n-1} \phi(x) \frac{\phi'(x) \phi(x)}{n} \right) dx
\]

\[
= \left[ \frac{(n-1)}{n} y^{\frac{1}{n-1}} \right]_{y=\phi(0)=0}^{y=\phi(x_0)=n} = \frac{1}{n} (n-1)(n^{n-1}) = 1 = \phi(x_0).
\]

Grandits and Falkner (2000) show that this condition is also sufficient for the existence of a finite stopping strategy inducing the final distribution. □

Thus, we have verified the existence of the equilibrium distribution derived in the previous section and thereby established the unique equilibrium outcome of the game.

### 4.3 An Extension: Asymmetric Starting Values

In this extension, we allow for heterogeneity in the starting values. In order to get an analytical solution, we restrict attention to the two-player case.

\(^{2}\)In the case \(\mu = 0\), the proof is identical with \(\phi(x) = \frac{x}{x_0}\)
Assume without loss of generality \( x_0^1 > x_0^2 \). The proof of the following proposition is similar to the construction of Proposition 1 and relegated to the appendix.

**Proposition 4.** In equilibrium, the cdf of the first player is given by

\[
F^1(x) = \min \left\{ 1, \frac{1}{2} \frac{\exp\left(-2\mu x \sigma^2\right) - 1}{\exp\left(-2\mu x_0^1 \sigma^2\right) - 1} \right\}.
\]

The cdf of the second player is given by

\[
F^2(x) = \min \left\{ 1, \rho + (1 - \rho) \frac{1}{2} \frac{\exp\left(-2\mu x \sigma^2\right) - 1}{\exp\left(-2\mu x_0^1 \sigma^2\right) - 1} \right\}
\]

where \( \rho \) is the probability of being absorbed in zero when playing until zero or \( x_0^1 \) is reached, i.e.

\[
\rho = \frac{\exp\left(-2\mu (x_0^1 - x_0^2) \sigma^2\right) - 1}{\exp\left(-2\mu (x_0^1 - x_0^2) \sigma^2\right) - \exp\left(2\mu x_0^2 \sigma^2\right)}.
\]

**Proof.** Feasibility of the cdf of player 1 is established in Proposition 2. For player 2, consider the following strategy: First, play until \( X^2_t \in \{0, x_0^1\} \). Then use the same stopping strategy strategy as player 1 if reaching \( x_0^1 \). Obviously, this induces the above cdf. By Doob’s optional stopping theorem, there is no profitable deviation as the stopped processes \( u^i(X^i_t) \) are supermartingales. Uniqueness of the equilibrium is shown in the appendix.

It is immediate from the equilibrium construction that the player with the lower starting value engages relatively more in gambling activity. Intuitively, he has to make up for his initial disadvantage by taking higher risks.
5 Features of the Equilibrium

5.1 Comparative Statics

This section carries out a comparative static analysis of the equilibrium in the case of symmetric starting values. In particular, we are interested in changes of the expected value of the stopped process induced by changes in the parameters. Expected value minus starting value can be interpreted as the expected revenue per player to the principal.

First we calculate the density from the cdf in Proposition 1

\[ f(x) = F'(x) = \frac{2\mu}{n(n-1)\sigma^2} \sqrt{\frac{\exp\left(-\frac{2\mu x}{\sigma^2}\right)}{n(1+\exp\left(-\frac{2\mu x_0}{\sigma^2}\right)) \left(1-\exp\left(-\frac{2\mu x_0}{\sigma^2}\right)\right)}}. \]

In the sequel, we derive a tractable closed-form solution for the expected value in the two-player case to convey the main intuition. The general formula for \( n \) players is stated in the appendix. Using the density \( f \) we calculate the expected value of the stopped process for \( n = 2 \)

\[ \mathbb{E}(X^i_{\tau^i}) = \mathbb{E}_f(x) = \int_0^x x f(x) dx = \frac{\sigma^2}{2\mu} + \frac{\exp\left(\frac{2\mu x_0}{\sigma^2}\right)}{2(1-\exp\left(\frac{2\mu x_0}{\sigma^2}\right))} (x_0 - \frac{\sigma^2 \ln\left(1+\frac{2\mu x_0}{\sigma^2}\right)}{2\mu}). \]

We begin by stating a result which directly follows from the fact that players never choose to stop their process immediately with probability one:

**Lemma 5.** \( \mu < 0 \implies \mathbb{E}(X^i_{\tau^i}) < x_0, \mu > 0 \implies \mathbb{E}(X^i_{\tau^i}) > x_0. \)

Hence, if times are bad, the principal (who might not be perfectly informed about the drift) incurs losses from holding a contest. The following limit results help to understand the effects of changes in \( \mu \) and \( \sigma \) more precisely. The proofs are omitted as they are straightforward from applying l’Hôpital’s rule.

**Lemma 6.** If \( \mu \to 0 \), then \( \mathbb{E}(X^i_{\tau^i}) \to x_0. \)
Thus, the expected value of the martingale case is also the limit for the sub- and supermartingale case. If the drift is negative but very small in absolute value, the incurred losses of the principal are moderate. On the other hand, the following result shows that if the negative drift is very high in absolute value, the losses in terms of expected value also approach 0.

**Lemma 7.** If $\mu \to -\infty$, then $\mathbb{E}(X^t_{\tau_i}) \to x_0$.

Intuitively, it does not pay anymore to induce variance in the final distribution as the probability to lose money if playing approaches 1. Hence, the principal loses most money in expectation if the market is moderately bad.

A similar effect is obtained regarding changes in the variance.

**Lemma 8.** Assume $\mu < 0$. If $\sigma \to 0$, then $\mathbb{E}(X^t_{\tau_i}) \to x_0$.

A small variance implies that losses occur with a higher probability as the drift gets relatively more important. In the limit, players do not find it profitable to play anymore, as they lose almost surely. On the contrary, if the variance is extremely high, the drift term gets less important. In the limit players are able to replicate the distribution of the martingale case in which no losses occur.

**Lemma 9.** Assume $\mu < 0$. If $\sigma \to \infty$, then $\mathbb{E}(X^t_{\tau_i}) \to x_0$.

The previous results are summarized in the following proposition and illustrated for an example in Figure 2.

**Proposition 5.** The expected value of the stopped process is non-monotone in drift and variance.

Hence, an increase in the drift does not necessary lead to an increase in the expected value of the stopped process. Intuitively, if investment opportunities improve, realized equilibrium returns in the contest might decline. This surprising effect results from the equilibrium requirement to keep other players indifferent. An increase in the drift thereby leads to a larger variance of the final distribution. For some parameter values of $\mu$, this effect dominates the effect of decreased losses per time.
Figure 2: An example for $n = 2$, $x_0 = 100$. The drift $\mu$ is shown on the abscissa and the expected value of the stopped process $E(X_i^\tau)$ on the ordinate for different values of variance $\sigma$.

In the next paragraph, we establish another result highlighting the lack of robustness of relative performance mechanisms. More specifically, we analyze changes in the expected value for the case that both $\mu$ and $\sigma$ approach zero at the same time. It turns out that not only the equilibrium structure may change crucially, but there even arises a discontinuity in the expected value:

**Proposition 6.** The expected value is non-continuous for $(\sigma, \mu) \to (0, 0)$.

**Proof.** Note that $E(X_i^\tau)$ is constant if $\frac{\mu}{\sigma^2}$ is constant. Consequently if $\mu$ and $\sigma$ go to zero and $\frac{\mu}{\sigma^2}$ is kept constant $E(X_i^\tau)$ does not converge to $X_0^i$.  

Hence, even if randomness almost disappears and $\mu < 0$, the principal cannot be guaranteed to bear no losses (which would the case for $\sigma = 0$). Summarizing, the previous results have emphasized problems of relative performance pay.
5.2 A Comparison to Related Models

The predictions of the model are in clear contrast to those of the static model in Lazear and Rosen (1981). They find that contests are suitable to induce efficient effort choice (the decision for how many periods to play in our model). This difference is due to a different information structure. If players had to specify the time period they play without receiving any additional information and \( n = 2 \), nobody would be willing to participate in a gamble with negative expected value as in Lazear and Rosen (1981).

In terms of predictions about equilibrium distributions, the present paper shows remarkable similarities to the literature on all-pay auctions with complete information (e.g. Hillman and Samet, 1989 or Baye et al., 1996).\(^3\) This may seem surprising at first sight because players optimize indirectly via a stochastic process here and bear no costs to play compared to everybody paying his bid in the auction. The simple common intuition driving both results is that the joint distribution of all other players has to make each player indifferent. However, differing from the all-pay auction, each player participates actively in the game in any equilibrium.

6 Conclusion

We have presented a continuous time model of a contest in which players can stop a stochastic process conditioning on previous realizations. It has been found that agents are willing to engage in socially wasteful gambling activity even if the market goes down. According to the model predictions, expected losses are particularly high if times are moderately bad. Furthermore, even if a principal is only slightly mistaken about the trend of a market he might incur large losses resulting from a discontinuity of the equilibrium structure if trend and variance approach zero.

Thus, we have given a theoretical argument against relative performance payments for CEOs to explain the puzzling paucity of this mechanism. Fur-

\(^3\)Complete information about valuations in the all-pay auction framework corresponds to complete information about starting values in this paper.
thermore, for mutual fund companies which are in permanent competition for cash inflow in the next period, the model predicts a large waste of money in times of moderately decreasing markets.

From a technical point of view, this paper has introduced a solution method for continuous time contest models based on recent contributions in probability theory. This approach may enhance future research as it provides a general framework which allows to solve contest models analytically via ODE’s and to disentangle how the choice of a particular contest success function influences the results.

7 Appendix

Proof of Lemma 1: The proof goes by contradiction. Assume there exists an agent \( j \) and a point \( x > 0 \) s.t. \( \mathbb{P}(X^j_{\tau_j} = x) > 0 \).

(i) If any other agent \( i \neq j \) ever reaches \( x \) he can play the strategy to stop only if \( X^i_t = y < x \) or \( X^i_t = x + \epsilon \), which yields an expected payoff of

\[
\mathbb{P}(X^i_{\tau_i} = y)u^i(y) + \mathbb{P}(X^i_{\tau_i} = x + \epsilon)u^i(x + \epsilon) .
\]

Taking the limit \( \epsilon \to 0 \)

\[
\lim_{\epsilon \to 0} (1 - \mathbb{P}(X^i_{\tau_i} = x + \epsilon)) u^i(y) + \mathbb{P}(X^i_{\tau_i} = x + \epsilon)u^i(x + \epsilon) = \lim_{\epsilon \to 0} u^i(x + \epsilon) .
\]

As \( \mathbb{P}(X^j_{\tau_j} = x) > 0 \) we get \( u^i(x) < \lim_{\epsilon \to 0} u^i(x + \epsilon) \). By continuity of the probability to reach the point \( x + \epsilon \) there exists an open interval \((z, x)\) such that for any \( X^i_t \in (z, x) \) the strategy to stop only if \( X^i_t \in \{y, x + \epsilon\} \) yields a payoff strictly higher then \( u^i(X^j_t) \). Thus, there exists an interval \((x - \epsilon, x)\) such that in equilibrium no agent \( i \neq j \) stops in the interval. But then agent \( j \), when arriving at \( x \), is strictly better off to continue to play until a point in \((x - \epsilon, x)\) or \( \max_{i \neq j} \pi^i \) is reached, which contradicts the equilibrium assumption.

(ii) If no other player ever reaches the interval \((z, x)\), a similar argument
shows that player $j$ is better off stopping before $x$ is reached.

Finally, assume that $\mathbb{P}(X^i_{r_i} = 0) > 0 \ \forall i$. Then, by the discontinuity in 0, there exists $\epsilon > 0$ such that each agent prefers to stop his process at $\epsilon$ which ensures winning with positive probability setting up the equilibrium assumption.

**Proof of Lemma 3:** Assume, by contradiction, there exists an interval $(x - \epsilon, x + \epsilon) \in [x, \overline{x}]$ such that $u^i(x)$ is constant. By $u^i(x) = \prod_{j \neq i} F^j(x)$, no agent $j \neq i$ stops in the interval $(x - \epsilon, x + \epsilon)$.

If no player $j \neq i$ stops above $x$, player $i$ would have been better off stopping at $\max_{j \neq i} \bar{x}_j$ which ensures winning with probability one (clearly, player $i$ always stopping above $\max_{j \neq i} \bar{x}_j$ cannot be an equilibrium, because each player can reach any finite value with probability greater than 0).

However, if there exist a player $j \neq i$ stopping above $x + \epsilon$, player $i$ will never stop on the interval $(x - \epsilon, x + \epsilon)$, because playing until $X^i_t \in \{x - \epsilon, \max_{j \neq i} \bar{x}_j\}$ guarantees him a strictly higher payoff.

Thus, no agent stops on $(x - \epsilon, x + \epsilon)$, but at least two agents stop above it. Take the infimum of the stopping values above $x + \epsilon$. The stopping agent would be better off continuing at this value (and by continuity on a small neighborhood around it) until $X^i_t \in \{x - \epsilon, \max_{j \neq i} \bar{x}_j\}$ which contradicts the equilibrium assumption.

**Proof of Lemma 4:** First assume, by contradiction, that there exists a point $x \in (\underline{x}^i, \overline{x}^i)$ such that $\mathbb{E}(d u^i(x)) \neq 0$. By continuity of $u^i$ above $x = 0$, there exists an interval $(x - \epsilon, x + \epsilon)$ such that $\mathbb{E}(d u^i(x)) \neq 0 \ \forall x \in (x - \epsilon, x + \epsilon)$.

On the other hand, by optimality of stopping at $X^i_t \in \{\underline{x}^i, \overline{x}^i\}$, there exists $x \in (\underline{x}^i, \overline{x}^i)$ s.t. $\mathbb{E}(d u^i(x)) > 0$. However, stopping at a point in $x \in (\underline{x}^i, \overline{x}^i)$ implies that there exists a point $\tilde{x} \in (\underline{x}^i, \overline{x}^i)$ s.t. $\mathbb{E}(d u^i(\tilde{x})) < 0$, because otherwise it would be strictly better to continue at all interior points until $X^i_t \in \{x^i, \overline{x}^i\}$. By continuity, player $i$ does not stop on a small interval around $\tilde{x}$. Hence, no player ever stops at a small interval around $\tilde{x}$ as $\mathbb{E}(d u^i(\tilde{x})) < 0 \implies \mathbb{E}(d u^j(\tilde{x})) < 0 \ \forall j$. This is contradiction to Lemma 2.

**Proof of Lemma 5:** First assume, by contradiction, that $\max_i \underline{x}^i \neq 0$. Then all players with $X^i_t < \max_i \underline{x}^i$ would have a strict incentive to continue play-
ing until $X^j_t \in \{0, \max_i \bar{x}^i\}$, because stopping below $\max_i \bar{x}^i$ means losing for sure. By continuity of $u^i$, this applies also for a neighborhood of $\max_i \bar{x}^i$ which contradicts Lemma 3.

Regarding $\bar{x}$, assume by contradiction that in equilibrium player $j$ has $\bar{x}^j < \max_i \bar{x}^i$. An agent $i$ playing above $\bar{x}^j$ with positive probability has to be indifferent at and above $\bar{x}^j$ by Lemma 3 ($\mathbb{E}(du^i(\bar{x}^j)) = 0$). Compared to the indifference condition of agent $i$ at $\bar{x}^j$, agent $j$ has the additional possibility of outperforming agent $i$ which makes him strictly better off playing at $\bar{x}^j$. □

Proof of Proposition 4. To prove uniqueness, note that Lemma 1-4 do not rely on any symmetry arguments and do still hold. Hence, the equation $u^i(x) = F^j(x)$ fixes the above construction uniquely given the right endpoint. However, by the minmax property (zero-sum game) each player must receive the same payoff as above in any equilibrium. By indifference between stopping immediately and playing until $X^j_t \in \{0, \bar{x}\}$, it follows directly that $\bar{x}$ is uniquely determined. By Lemma 1 only one agent might set a mass point at 0. Feasibility implies that the agent with the lower starting value sets the mass point at zero and uniquely determines the size of the mass point. □

Formula for the Expected Value in the n-Player Case:

$$\mathbb{E}(x) = \int_0^\bar{x} xf(x)dx = (\bar{x}F(\bar{x}) - 0F(0)) - \int_0^\bar{x} F(x)dx$$

$$= \bar{x} - \int_0^\bar{x} n^{-1} \sqrt{\frac{1}{n} \exp(-2\mu x) - 1} dx$$

$$= \bar{x} + \frac{n^{-\sqrt{1 - \exp(-2\mu \bar{x})}}}{2\mu} (n - 1) Hyp\left(\frac{1}{n - 1}, \frac{1}{n - 1}, \frac{n - 2}{n - 1}, \exp(2\mu \bar{x})\right).$$

$Hyp$ denotes the Gauss hypergeometric function. Note that for the special case of $n = 3$ an analytic formula for the expected value not involving the Gauss hypergeometric function can be obtained.
References


