Portfolio Optimization and Rank Dependent Expected Utility

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Abstract

This paper provides for the first time the solution to the static portfolio optimization problem in the Rank Dependent Expected Utility (RDEU) framework. The optimal portfolio is determined for quite general utility functions, probability weighting distortions and insurance constraints. The role of the probability weighting transformation is analyzed. It is proved that the monotonicity of the ratio, which is equal to the price weighting density divided by the behavioral weighting density, determines crucially the optimal profile. This result explains some market anomalies that are observed on financial structured products.

Key Words: portfolio optimization, rank dependent utility, probability weighting function.

JEL classification: C61, G11, L10.

1 Introduction

Standard decision theory assumes that individuals maximize their expected utilities (ES). Nevertheless, many empirical and theoretical studies have been developed against the expected utility theory (see e.g. Allais, 1953; Kahneman and Tversky, 1979; Epstein and Schneider, 2003). The expected utility theory is fundamentally based on the Independence Axiom. As shown by Allais (1953), this axiom may fail empirically and can induce some paradoxes. Additionally, as proved by Cohen and Tallon (2000), the expected utility theory implies that the utility function must simultaneously represent the choice among possible outcomes, and also model the attitude towards risk. Thus, for example, an investor who has a decreasing marginal utility is necessarily risk-averse. During
the 80’s, alternative theories have been introduced by slightly modifying or re-laxing the original axioms. For example, the Weighted Utility Theory of Chew and MacCrimmon (1979) who consider a weaker axiom of independence; the Prospect Theory of Kahnemann and Tversky (1979, 1992); the Non-Linear Expected Utility Theory of Machina (1982a, 1982b); the Anticipated Utility Theory of Quiggin (1982); the Dual Theory of Yaari (1987).

Quiggin (1982) has assumed that individuals can modify the objective probability distribution. The notion of probability weighting transformation has been also considered by Tversky and Kahnemann (1992). This probability distortion is usually observed from empirical analysis. The determination of the probability weighting function, in particular its curvature, has been studied by Wu and Gonzalez (1996) and Prelec (1998). When objective probabilities are distorted or unknown, the estimation of the utility function is a more involved problem. In this framework, Wakker and Deneffe (1996) have provided the elicitation of von Neumann-Morgenstern utilities. Abdellaoui (2000) and de Palma, Picard and Prigent (2007) have proposed various methods to estimate either the weighting function either the utility function (see also Abdellaoui et al., 2008).

The expected utility approach has been widely developed in the financial portfolio theory, since it was the common assumption to model investor choices. Additionally, analytical results can be deduced in this framework for standard cases as proved by Merton (1969). Such results have been further extended for example when taking account of market incompleteness, labor income, stochastic horizon...(for a survey, see e.g. Karatzas and Shreve, 1998; Campbell and Viceira, 2002; Prigent, 2007). Most of these results have been established in the continuous-time setting. Karatzas et al. (1986) and Cox and Huang (1989) have determined the optimal strategies of an investor maximizing the expected utility of her consumption and portfolio value, when assets prices are assumed to be diffusion processes. Cvitanic and Karatzas (1996) have solved this problem with constraints on portfolio.

However, actual portfolios strategies correspond to discrete-time trading, in particular in a one-period setting. The determination of static optimal portfolio is called the optimal positioning problem. It has been introduced by Leland (1980), and Brennan and Solanki (1981). The portfolio value is a function of a given benchmark. Its payoff maximizes the investor’s expected utility. It is characterized by the investor’s risk aversion and involves option-based strategies. Carr and Madan (2001) show that assuming the existence of out-of-the-money European puts and calls of all strikes allows the determination of the optimal positioning in a complete market. This is the counterpart of the assumption of continuous trading. This assumption is justified when there is a large number of option strikes (e.g. the S&P500, for example). Introduced by Leland and Rubinstein (1976), the portfolio insurance theory often considers portfolio payoffs which are functions of a benchmark (a specified portfolio of common assets). At maturity, the investor can limit downside risk while participating

2 The Option Based Portfolio Insurance (OBPI), introduced by Leland and Rubinstein
in upside markets. Investors can receive a given percentage of their initial capital, even in falling markets. However, more specific insurance constraints can be introduced, for example for institutional investors (see e.g. Bertrand, Lesne and Prigent (2001) for quite general insurance constraints).  

In this paper, for the first time, the optimal positioning problem is solved for the broad class of rank dependent expected utilities (RDEU). The optimal portfolio is determined for quite general utility functions and probability weighting functions. Additionally, insurance constraints can also be taken into account. The influence of the probability weighting transformation is examined. The main result is that the optimal portfolio profile is a function of the ratio of the asset pricing weighting function upon the behavioral weighting one. Therefore, the monotonicity of this ratio determines crucially the optimal portfolio. This property extends previous results about the optimal positioning, from the standard expected utility case to the rank dependent expected utility case. It can be applied to explain how to buy or sell derivative assets according to behavior towards risk. Therefore, it provides an explanation of some market anomalies when looking at financial structured products. For instance, portfolio profile is not always increasing with respect to the benchmark. This result can also allow the determination of the compensating variations (certainty equivalents). These latter ones are introduced to measure the adequacy of a given portfolio to the investor preferences, as introduced by De Palma and Prigent (2008, 2009).

This paper is organized as follows. In Section 2, some basic notions about RDEU are recalled. In section 3, optimal portfolios are determined as functions of a given benchmark, in the RDEU framework, which provides an extension of Carr and Madan (2001) and Prigent (2006). The insured or not insured optimal portfolio is characterized for arbitrary utility functions and probability weighting functions. Results about Nemitski functional, subdifferentiability (see Ekeland and Turnbull, 1983) and concavity in global optimization (see Li et al., 2001) are applied to prove some of these results. A generalization of the Kuhn-Tucker theorem to the infinite-dimensional case is also used to solve the insurance problem.

Basic examples are examined in Section 4, in particular for the anticipated utility (Quiggin, 1982) and the cumulative prospect theory (Tversky and Kahneman, 1992). For each of these cases, this paper provides the corresponding optimal positioning of options such as puts or straddles. Such results are in line with some previous empirical observations. For example, contrary to the standard expected utility case, the optimal payoff may be no longer an increasing function of the risky asset, but, for instance, may correspond to a long or short position on a straddle. Some of the proofs are gathered in Appendix.

(1976), consists of a portfolio invested in a risky asset $S$ (usually a financial index such as the S&P) covered by a listed put written on it. Whatever the value of $S$ at the terminal date $T$, the portfolio value will be always greater than the strike $K$ of the put.

3See also El Karoui et al. (2005) who determine the optimal portfolio with an American capital guarantee.
2 Rank dependent expected utility theory

2.1 Introduction

One of the first theories consistent with the Allais paradox is the “weighted utility theory”, introduced by Chew and MacCrimmon (1979) and further developed by Chew (1989) and Fishburn (1983). The idea is to apply a transformation on the initial probability.

The basic result of Chew and MacCrimmon yields to the following representation of preferences over lotteries \( L = \{(\omega_1, p_1), ..., (\omega_m, p_m)\} \):

\[
U(L) = \sum_i u(x_i)\phi(p_i) \quad \text{with} \quad \phi(p_i) = p_i / \left( \sum_i v(x_i)p_i \right),
\]

where \( u \) and \( v \) are two different elementary utility functions.

Another approach is the “prospect theory” introduced by Kahneman and Tversky (1979). The idea of the prospect theory is to represent references by means of a function \( \phi \) such that the utility of a lottery \( L = \{(x_1, p_1), ..., (x_n, p_n)\} \) is given by:

\[
U(L) = \sum_i u(x_i)\phi(p_i),
\]

where \( \phi \) is an increasing function defined on \([0, 1]\) with values in \([0, 1]\) and \( \phi(0) = 0, \phi(1) = 1 \).

The function \( \phi(\cdot) \) is a transformation of the initial probability and corresponds to a decision weight functional. It allows to take account of a “certainty effect”. For example, if the function \( \phi \) is not left-continuous at 1, then \( \phi(p) < p \) maybe in a neighborhood of 1. This is the result of the passage from certitude to uncertainty. Note that equality \( \sum_{i=1}^n \phi(p_i) = 1 \) may no longer be true.

Using this transformation, the Allais paradox can be solved. Additionally, from experimental observations, Tversky and Kahneman (1992) argue that it is necessary to distinguish positive results (gains) from negative ones (losses) from experimental observations. However, the sub-additivity of \( \phi \) which is induced:

\[
\forall p_1, p_2 \in [0, 1[, \quad \phi(p_1) + \phi(p_2) < \phi(p_1 + p_2),
\]

may imply the violation of the first-order stochastic dominance, as well as other models with weighted probabilities. To solve this problem, alternative approaches can be proposed.
The “Rank Dependent Expected Utility” theory (RDEU) assumes that people consider cumulative distribution functions rather than probabilities themselves. In this framework, we can introduce preference representations which are compatible with the first-order stochastic dominance.

The functional representation of preferences is defined as follows:

**Definition 1** For all random variables \( X \) and \( Y \) which model results or consequences and with values in \([-M,M]\), we have:

\[
X \succ Y \Leftrightarrow V(X) \geq V(Y), \text{ with } V(Z) = \int_{-M}^{M} u(z)d\Phi(F_Z(z)), \tag{4}
\]

where function \( u(\cdot) \) is continuous and differentiable, non-decreasing and unique up to a non-negative linear transformation. The function \( \Phi(\cdot) \) is continuous and non-decreasing from \([0,1]\) into \([0,1]\). The function \( F_Z(.) \) denotes the cumulative distribution function (cdf) of \( Z \). Without loss of generality, it can be assumed that if \( \Phi(0) = 0 \) and \( \Phi(1) = 1 \), then \( \Phi(\cdot) \) is unique.

Note that for a discrete lottery \( L = \{(x_1, p_1), \ldots, (x_m, p_m)\} \) with \( x_1 \leq x_2 \leq \cdots \leq x_n \),

the utility \( V \) is given by:

\[
V(L) = \sum_{i=1}^{n} u(x_i) \left[ \Phi(\sum_{j=1}^{i} p_j) - \Phi(\sum_{j=1}^{i-1} p_j) \right], = u(x_1) + \sum_{i=2}^{n} (u(x_i) - u(x_{i-1})) \left[ 1 - \Phi(\sum_{j=1}^{i} p_j) \right]. \tag{5}
\]

Since the weights \( \Phi(\sum_{j=1}^{i} p_j) \) depend on the ranking of the outcomes \( x_i \), this preference representation is called “rank dependent expected utility.” These weights are determined by first ranking outcomes from the worst to the best then by summing up the utilities weighted by the sequence \( (\Phi(\sum_{j=1}^{i} p_j) - \Phi(\sum_{j=1}^{i-1} p_j)) \). Thus, an objective probability is assumed to exist, but individuals transform this given probability distribution by using a transformation of its cdf. This allows the first-order stochastic dominance.

**Remark 2** The RDEU is a generalization of the expected utility criterion (EU). Indeed, for \( \Phi(p) = p, \forall p \in [0,1] \), the functional representation of preferences is given by:

\[
V(L) = \sum_{i=1}^{n} u(x_i) \left[ \sum_{j=1}^{i} p_j - \sum_{j=1}^{i-1} p_j \right] = \sum_{i=1}^{n} p_i u(x_i) = EU(L).
\]

If \( \Phi \) is not the identity but \( u(x) = x \), then the RDEU is the dual theory of Yaari (1987).
As mentioned in Tallon (1997), the RDEU has several advantages. Contrary to the EU, the RDEU allows separation of the behavior towards wealth from the behavior towards risk. Thus, the RDEU is compatible with usual empirical observations which show that individuals under- or overestimate probabilities of random events (i.e., are either pessimistic or optimistic). Contrary also to the EU, the RDEU allows identification of two notions of risk-aversion: the standard weak risk-aversion of Arrow-Pratt and the strong risk-aversion of Rothschild and Stiglitz (1970, 1971). Tallon (1997) proves that strong risk-aversion allows the interpretation of RDEU: the beliefs of an individual are characterized by a given set of probability distributions and her utility is the infimum of the expectations of her utility on this set.

Several models have been proposed in the RDEU framework, as recalled in what follows.

2.2 Anticipated utility theory

Quiggin (1982) keeps three main properties of the ES theory: the transitivity, the first-order stochastic dominance, and the continuity. However, he adds the following axiom:

**Definition 3 (Weak independence Axiom)**

Consider two lotteries

\[ L_X = \{(x_1, p_1), \ldots, (x_n, p_n)\} \text{ and } L_Y = \{(y_1, p_1), \ldots, (y_n, p_n)\}, \]

such that

\[ x_1 \leq \ldots \leq x_n \text{ and } y_1 \leq \ldots \leq y_n, \]

and

\[ \forall i \in \{1, \ldots, n\}, P[X=x_i] = P[Y=y_i]. \]

Assume that there exists a common value \( x_{i_0} = y_{i_0} \). Consider two lotteries \( L_{X_0} \) and \( L_{Y_0} \) which are equal respectively to \( L_X \) and \( L_Y \), except that \( x_{i_0} \) and \( y_{i_0} \) are replaced by another common value.

The preference \( \succeq \) is weak independent if and only if:

\[ L_X \succeq L_Y \iff L_{X_0} \succeq L_{Y_0}. \quad (6) \]

**Remark 4** Consider a lottery \( L = \{(x_1, p_1), \ldots, (x_n, p_n)\} \). Then a functional \( V \) which satisfies Quiggin’s conditions is given by:

\[ V(L) = \sum_{i=1}^{n} u(x_i) \left[ \Phi \left( \sum_{j=1}^{i-1} p_j \right) - \Phi \left( \sum_{j=1}^{i} p_j \right) \right], \quad (7) \]

where \( \Phi \) is non-decreasing on \([0, 1]\) to \([0, 1]\), and is concave on \([0, \frac{1}{2}]\), \( \Phi(p_i) > p_i \) and convex on \([\frac{1}{2}, 1]\), \( \Phi(p_i) < p_i \) with \( \Phi(\frac{1}{2}) = \frac{1}{2} \) and \( \Phi(1) = 1 \).
As proposed in Quiggin (1982), the function $\Phi$ can be chosen as follows:

$$\Phi(p) = \frac{p^\gamma}{(p^\gamma + (1 - p)^\gamma)^{\frac{1}{\gamma}}}.$$

(8)

with, for example, $\gamma = 0.6$.

![Fig 1. The probability weighting function](image)

**Remark 5** Under the previous assumptions, the first-order stochastic dominance is satisfied. Additionally, the Allais paradox is solved and the model is in accordance with the empirical result of Tversky and Kahneman (1992): individuals weight more events with small probabilities and weight less those with high probabilities. Such preference representation implies that no investor will diversify; in the presence of one riskless asset and one risky asset, either the investor buys only the riskless one, or only the risky asset. However, when both assets are risky, and if a “background risk” (such as illness, accident, fire...) exits, diversification can be observed, as shown by Eeckhoudt (1997).

Many empirical experiences have proved that individuals do not have the same attitude towards losses and gains. The utility on losses appears to be convex, whereas the utility on gains appears to be concave. The value of each component is determined by taking the expected utility with respect to distortions of the distribution function which may differ for the positive and the negative parts of the distribution. This model can be viewed as a generalization of the standard rank-dependent utility model, for which the same distortion function is used for the whole distribution. This kind of behavior is modelled by the “Cumulative Prospect Theory”.
2.3 Cumulative prospect theory

Tversky and Kahneman (1992) have introduced both specific utility functions for losses and gains and a transformation function of the cumulative distributions. There exist two functions, \( w^- \) and \( w^+ \) defined on \([0, 1]\), and an utility type function \( v \) such that the utility \( V \) on the lottery \( L = \{(x_1, p_1), \ldots, (x_n, p_n)\} \) with \( x_1 < \ldots < x_m < 0 < x_{m+1} < \ldots < x_n \) is defined as follows: define \( \Phi^- \) and \( \Phi^+ \) by: \( \Phi^- = w^-(p_1) \) and \( \Phi^+ = w^+(p_n) \).

\[
\begin{align*}
\Phi^-_i &= w^- \left( \sum_{j=1}^i p_j \right) - w^- \left( \sum_{j=1}^{i-1} p_j \right), \forall i \in \{2, \ldots, m\}, \\
\Phi^+_i &= w^+ \left( \sum_{j=1}^i p_j \right) - w^+ \left( \sum_{j=i+1}^n p_j \right), \forall i \in \{m+1, \ldots, n\}.
\end{align*}
\]  

Then, \( V \) is given by: \( V(L) = V^-(L) + V^+(L) \) with

\[
V^-(L) = \sum_{i=1}^m v(x_i) \Phi^-_i \quad \text{and} \quad V^+(L) = \sum_{i=m+1}^n v(x_i) \Phi^+_i.
\]  

When the probability distribution \( F \) has a pdf \( f \) on \([-M, M]\), and the functions \( w^- \) and \( w^+ \) have derivatives \( w^-' \) and \( w^+ '\), then:

\[
V(L) = \int_{-M}^{0} v(x) w^-(F(x)) f(x) dx + \int_{0}^{M} v(x) w^+(1-F(x)) f(x) dx.
\]

As in Quiggin (1982), both functions \( w^- \) and \( w^+ \) can be chosen as follows: \( w(p) = \frac{F^\gamma}{(p^{\gamma}+(1-p^{\gamma})^{\gamma+1})^{\gamma}} \), with, for example, \( \gamma^- = 0.69 \) and \( \gamma^+ = 0.61 \).

We consider probability distributions associated to lotteries having density functions. In what follows, we provide general results assuming that there exists a functional \( \Phi \) which modifies the historical probability, taking account of the investor’s behavior. When basic assets are assumed to have pdf, another functional \( \varphi \), defined on the set of probability density functions (pdf), can be associated to the functional \( \Phi \). The function \( \varphi \) satisfies the following conditions:

Assumption 1: for any pdf \( f \), \( \varphi(f) \) is positive.

Assumption 2: for any pdf \( f \), \( \int \varphi(f)(x) dx = 1 \).

Assumption 3: for any probability distribution \( F \) with pdf \( f \), the distribution \( \Phi[F] \) has a pdf equal to \( \varphi(f) = f \psi(F) \), where \( F \) denotes the cdf associated to \( f \).

**Proposition 6** These assumptions are satisfied for the expected utility maximization, and for both the anticipated utility theory and the cumulative prospect theory.

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4 See de Palma, Picard and Prigent (2007).

5 We can consider discrete probabilities as well. Indeed, standard weak convergence results (see for example basic Monte Carlo simulations) prove that any lottery with a pdf is the limit of sequences of discrete lotteries (see de Palma et al. (2007), for more details).
Proof. For the first case, we consider \( \varphi(f) = f \) (i.e. \( \psi(F) \equiv 1 \)).

For the second case, due to the modification of probability weights (see Quiggin, 1982), the pdf \( \varphi(f) \) is equal to \( f w[F(.)] \), where the function \( w \) can be chosen as in Relation (8).

For the third case, the modification of probability weights depend on the position relative to the given level \( v^* \). Noth functions \( w^- \) and \( w^+ \) can be chosen as previously.

There exists a level \( x^* \), two functions \( w^- \) and \( w^+ \) defined on \([0,1] \) with derivatives \( w^-0 \) and \( w^+0 \) such that the individual maximizes:

\[
\int_a^v U(v) w^-F(F(v))f(v)dv + \int_v^b U(v) w^+1-F(F(v))f(v)dv. \tag{12}
\]

with for example \( \gamma^- = 0.69 \) and \( \gamma^+ = 0.61 \).

Then, the functional \( \varphi(f) \) is defined by \( \varphi(f)(v) = f(v)\psi(F)(v) \) with

\[
\psi(F)(v) = \frac{w^-F(v)1_{v \leq v^*} + w^+1-F(v)1_{v \geq v^*}}{w^-F(v^*) + w^+1-F(v^*)}. \tag{13}
\]

3 Optimal payoffs as functions of a benchmark

3.1 The financial market

This subsection extends previous results about expected utility maximization of Leland (1980), Brennan and Solanki (1981), Carr and Madan (2001) and Prigent (2006) to the rank dependent expected utility case. Suppose that the investor maximizes an expectation of her utility \( U \) with possible modification of probabilities. She is a price taker (for example, her benchmark \( S \) is the SP&500 and her investment is too weak to modify the index value). This framework is quite general: it includes for example the weighted utility introduced in Chew (1989), the rank-dependent utility (see Segal, 1989) or the cumulative prospect theory of Kahneman and Tversky (1992).

We assume the existence of three basic financial assets: the cash associated to a discount factor \( N \), the bond \( B \) and the stock \( S \) (a financial index for example). We suppose that the investor determines an optimal payoff \( h \) which is a function defined on all possible values of the assets \((N_T, B_T, S_T)\) at maturity \( T \). If the market is complete, this payoff can be achieved by the investor. The market can be complete for example if the financial market evolves in continuous-time and all options can be dynamically duplicated by a perfect hedging strategy. It can be still complete if for example, in one period setting, European options of all strikes are available on the financial market. In this setting, the inability to trade continuously potentially induces investment in cash, asset \( B \), asset \( S \) and all European options with underlying assets \( B \) and \( S \) (if cash and bond are non
stochastic, only European options on $S$ are required). The market can be also incomplete. In that case, the solution given in this section is only “theoretical” but still interesting to know since the optimal payoff can be approximated by investing on traded assets (in practice, the investor defines an approximation method, which may take transaction costs or liquidity problems into account).

Under the standard condition of no-arbitrage, the assets prices are calculated under risk neutral probabilities. If markets exist for out-of-the-money European puts and calls of all strikes, then it implies the existence of an unique risk-neutral probability that may be identified from option prices (see Breeden and Litzenberger, 1978). Otherwise, if there is no continuous trading, generally the market is incomplete and a one particular risk-neutral probability $Q$ is used to price the options. It is also possible that stock prices change continuously but the market may be still dynamically incomplete. Again, it is assumed that one risk-neutral probability is selected. Assume that prices are determined under such measure $Q$. Denote by $\frac{dQ}{dP}$ the Radon-Nikodym derivative of $Q$ with respect to the historical probability $P$. Denote by $N_T$ the discount factor and by $M_T$ the product $N_T \frac{dQ}{dP}$.

Due to the no-arbitrage condition, the budget constraint corresponds to the following relation:

$$V_0 = \mathbb{E}_Q[h(N_T, B_T, S_T)N_T] = \mathbb{E}_P[h(N_T, B_T, S_T)M_T].$$

(14)

### 3.2 The optimal portfolio profile

In what follows, the utility $U$ of the investor is supposed to be increasing and piecewise differentiable.

**Definition 7** If the marginal utility $U'$ is invertible, we denote by $J$ its inverse.

Introduce the “behavioral” probability $\tilde{P}$ equal to $\Phi[P]$, where $\Phi$ denotes the weighting functional which modifies the objective probability. The investor has to solve the following optimization problem:

$$\text{Max } \mathbb{E}_{\tilde{P}}[U(V_T)] \text{ under } V_0 = \mathbb{E}_P[V_TM_T].$$

(15)

The expectation $\mathbb{E}_{\tilde{P}}[U(V_T)]$ is equal to

$$\int U[v] f_V(v) \psi(F_V(v)) dv,$n(16)

where $\psi(.)$ denotes the density of the behavioral” probability $\tilde{P}$ with respect to the objective probability $P$.

For the optimal positioning, the portfolio value $V$ is a function of the basic assets: $V = h(N_T, B_T, S_T)$ and the investor solves:

$$\text{Max } \mathbb{E}_{\tilde{P}}[U[h(N_T, B_T, S_T)]] \text{ under } V_0 = \mathbb{E}_P[h(N_T, B_T, S_T)M_T].$$

(17)
Assume that \( X_T = (N_T, B_T, S_T) \) has a pdf denoted by \( f \). Define the transformation \( \psi_h \) by:

\[
\psi_h[F(x)] = \psi[F_h(h(x))],
\]

where \( F_h(h(x)) = \mathbb{P}[h(X_T) \leq h(x)] \).

Then, the expectation \( \mathbb{E}_g[U(h(N_T, B_T, S_T))] \) is equal to

\[
\int U[h(x)] f(x) \psi_h[F(x)] \, dx.
\]

To simplify the presentation of main results, we suppose as usual that function \( h \) fulfills:

\[
\int_{\mathbb{R}^+} h^2(x) f(dx) < \infty.
\]

It means that \( h \in L^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T}(dx)) \), which is the set of the measurable functions with squares that are integrable on \( \mathbb{R}^{+3} \) with respect to the distribution \( \mathbb{P}_{X_T}(dx) \).

A new functional \( \Gamma_U \) is associated to the utility function \( U \). It is defined on the space \( L^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T}(dx)) \) by:

\[
\Gamma_U(Y) = \mathbb{E}_{\mathbb{P}_{X_T}}[U(Y)].
\]

\( \Gamma_U \) is usually called the Nemitski functional associated with \( U \) (when \( U \) is concave, see Ekeland and Turnbull (1983) for definition and basic properties).

Denote also by \( g \) the density of \( M_T \) with respect to \( \mathbb{P} \). Assume that \( g \) is a function defined on the set of the values of \( X_T \) and \( g \in L^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T}(dx)) \). Then, the optimization problem is reduced to:

\[
\max_{h \in L^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T}(dx))} \int_{\mathbb{R}^+} U[h(x)] f(x) \psi_h[F(x)] \, dx,
\]

under \( V_0 = \int_{\mathbb{R}^+} h(x) g(x) f(x) \, dx \).

Denote by \( \tilde{\Gamma}_U \) the functional defined on the set \( L^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T}(dx)) \) by:

\[
\tilde{\Gamma}_U(h) = \int_{\mathbb{R}^+} U[h(x)] f(x) \psi_h[F(x)] \, dx.
\]

Let \( \Lambda \) be the linear functional such that

\[
\Lambda(h) = \int_{\mathbb{R}^+} h(x) g(x) f(x) \, dx.
\]

\[\text{For } r = 0, g \text{ is the Radon-Nikodym density } dQ/d\mathbb{P}.\]
Proposition 8 Assume that $\varGamma_U(.)$ is differentiable. Then, every relative maximum $h^*$ of $\varGamma_U$ under condition (21) must necessarily satisfy the first-order condition: there exists a scalar $\lambda$ such that

$$\frac{\partial \varGamma_U(h)}{\partial h} = \lambda \Lambda. \quad (24)$$

Proof. This property is a consequence of general results about optimization under constraints, when both the functional to optimize and the function characterizing the constraint are differentiable. Here, we note that the derivative of the continuous linear functional $L$ is equal to itself. Thus, we deduce the result.

Proposition 9 If the weighting function $\tilde{\psi}_h$ does not depend on $h$, then, every relative maximum $h^*$ of $\varGamma_U$ under condition (21) must necessarily satisfy the first-order condition: there exists a scalar $\lambda$ such that:

$$U'(h(x)) = \lambda \left( \frac{g(x)}{\tilde{\psi}(F(x))} \right). \quad (25)$$

Furthermore, if the utility $U$ is convex and the marginal utility is invertible, then the optimal payoff $h^*$ is given by:

$$h^* = J(\lambda g/\tilde{\psi}(F)), \quad (26)$$

where $\lambda$ is the scalar Lagrange multiplier such that

$$V_0 = \int_{\mathbb{R}^+} J(\lambda g/\tilde{\psi}(F))g(x)f(x)dx.$$

Proof. From the properties of the utility function $U$, the functional $\varGamma_U$ is differentiable (the Gâteaux-derivative exists) on $L^2(\mathbb{R}^+; \mathbb{P}_X)$. Additionally, the budget constraint is a linear function of $h$. Thus, there exists exactly one solution $h^*$ which is the solution of $\frac{\partial L}{\partial h} = 0$, where the Lagrangian $L$ is defined by:

$$L(h, \lambda) = \int_{\mathbb{R}^+} [U(h(x))]f(x)\tilde{\psi}[F(x)]dx + \lambda \left( V_0 - \int_{\mathbb{R}^+} h(x)g(x)f(x)dx \right), \quad (27)$$

where $\lambda$ is the Lagrange multiplier associated to the budget constraint.

The first-order condition is equivalent to:

$$\int_{\mathbb{R}^+} [U'(h(x))]f(x)\tilde{\psi}[F(x)]dx = \lambda \int_{\mathbb{R}^+} g(x)f(x)dx. \quad (28)$$

Thus: for any $x$,

$$[U'(h(x))]f(x)\tilde{\psi}[F(x)] = g(x)f(x),$$

from which, we deduce that $h^*$ satisfies: $U'(h^*) = \lambda g/\tilde{\psi}(F)$. Therefore, $h^* = J(\lambda g/\tilde{\psi}(F))$. 

\[\blacksquare\]
Remark 10 Under concavity assumption and if the marginal utility is invertible, the first-order condition provides a necessary and sufficient condition to get the (global) optimum $h^*$. The previous result also proves that the ratio of the risk-neutral density $g$ upon the density of the behavioral probability $\hat{\psi}(F)$ plays a key role on the determination of the optimal portfolio profile, as illustrated in next sections.

However, the utility function may be not concave. For instance, in Tversky and Kahneman (1992), the investor if loss averse. Thus, her utility function is convex on losses. Nevertheless, a characterization of the optimal solution can be provided by means of concavification. Li et al. (2001) propose a general method for the concavification of monotone functions.

Definition 11 Let $l : \mathbb{R}^n \rightarrow \mathbb{R}$ a function defined in a set $D \subset \mathbb{R}^n$. The function $l$ is said increasing (resp. decreasing) if for any $y, z \in D$ with for $y_i \leq z_i$ we have $l(y) \leq l(z)$ (resp. $l(y) \geq l(z)$).

Let $l : \mathbb{R}^n \rightarrow \mathbb{R}$ a twice differentiable function in a set $D \subset \mathbb{R}^n$. For any $z = (z_1, ..., z_n)^T \in \mathbb{R}^n$, denote by $z^{1/p}$ the vector $(z_1^{1/p}, ..., z_n^{1/p})^T$ and by $\nu_p(.)$ the function defined by $\nu_p(y) = y^p$.

Definition 12 A $p$-th power concavification transformation of $l$ is defined as follows:

$$
\phi_p(z) = -\left[l\left(z^{1/p}\right)\right]^{-p}.
$$

Lemma 13 Let $l : \mathbb{R}^n \rightarrow \mathbb{R}$ a twice differentiable function in a set $D \subset \mathbb{R}^n$. Suppose that there exist two non negative numbers $\epsilon_0$ and $\epsilon_1$ such that

$$
\begin{align*}
\epsilon_0, \forall y \in D, & \quad l(y) \geq \epsilon_0, \\
\frac{\partial l}{\partial y_j} \geq \epsilon_1, \forall y \in D \text{ and } \forall j \in \{1, ..., n\}.
\end{align*}
$$

Then there exists $p_1 > 0$ such that $\phi_p(.)$ is concave on the set $D_{p_1} = \nu_p(D)$.

Note that there exists a one to one correspondence between $y \in D$ and $z \in D_{p_1}$.

Lemma 14 Assume that $l$ satisfies condition (30), then it is equivalent to solve $\max_{x \in D} [l(x)]$ or to solve $\max_{x \in D_{p_1}} [\phi_l(y)]$.

We can apply such property to get a concavification of the objective function $\hat{\Gamma}_U(h) = \int_{\mathbb{R}^{k+3}} U[h(x)] f(x) \hat{\psi}[F(x)] dx$. Note for example, that when the set $\Omega$ of random events is finite ($|\Omega| = n$), $\hat{\Gamma}_U(h) = \sum_{j=1}^n U[h(x_j)] \hat{p}(x_j) \hat{\psi}[F(x_j)]$. Set $y_j = h(x_j)$ and $l(y_1, ..., y_n) = \hat{\Gamma}_U(h)$. Then, the concavification of $\hat{\Gamma}_U(.)$ is equivalent to the concavification of the utility function $U$ itself. This result can be extended to the infinite case.
Define the functional $\Gamma_{U}^{c,p}(.)$ by
\[ \Gamma_{U}^{c,p}(h) = -\int_{\mathbb{R}^{+3}} \left( U \left[ h(x)^{1/p} \right] \right)^{-p} f(x) \widehat{\psi} [F(x)] \, dx \]  
(31)

**Proposition 15** Assume that there exist two non negative numbers $\epsilon_0$ and $\epsilon_1$ such that
\[ U(v) \geq \epsilon_0, \forall v \in D, \quad \text{and} \quad \frac{\partial U}{\partial v} \geq \epsilon_1, \forall v \in D. \]  
(32)
Then there exists $p_1 > 0$ such that $\Gamma_{U}^{c,p}$ is concave.

**Corollary 16** If the utility function $U$ is twice differentiable and satisfies assumption (32), then the optimal solution $h^\ast$ is characterized by the first-order condition corresponding to $\Gamma_{U}^{c,p}$.

### 3.3 Properties of the optimal portfolio

Suppose for example that cash and bond are non stochastic. Then, the properties of the optimal payoff $h^\ast$ as function of the benchmark $S$ can be analyzed.

From Proposition (6), we can assume for example that $\widehat{\psi}_h[F(x)]$ has the following form:
\[ \widehat{\psi}_h[F(x)] = \widehat{\psi}^{-}_h[F(x)]1_{h(x) \leq v^\ast} \widehat{\psi}^{+}_h[F(x)]1_{h(x) > v^\ast}. \]  
(33)

Note that for the Quiggin case, we have $\widehat{\psi}^{-}_h = \widehat{\psi}^{+}_h$.

For the Tversky and Kahneman case, we get:
\[ \widehat{\psi}^{-}_h(F(x)) = \frac{w^{-}[F_h(h(x))]}{w^{-}[F_h(v^\ast)] + w^{+}[1 - F_h(v^\ast)]}, \]  
(34)
\[ \widehat{\psi}^{+}_h(F(x)) = \frac{w^{+}[1 - F_h(h(x))]}{w^{-}[F_h(v^\ast)] + w^{+}[1 - F_h(v^\ast)]}. \]  
(35)

Consider now the case where both $\widehat{\psi}^{-}_h$ and $\widehat{\psi}^{+}_h$ do not depend on $h$. For instance, we search the solution on a given subset $H$ of $L^2(\mathbb{R}^{+3}, P_{X_T})$ which is defined as follows: there exists an increasing sequence $(s_i)_{1 \leq i \leq n}$ such that $\forall i$, the payoff $h^\ast$ is monotone on $[s_i, s_{i+1}]$. The monotony on each $[s_i, s_{i+1}]$ is fixed. Such assumption is linked to the ranking of the values $h(s)$ on each interval $[s_i, s_{i+1}]$.

In that case, the determination of $F_h(h(s)) = P[h(S_T) \leq h(s)]$ is made as follows:
\[ \text{If } h \text{ is increasing, } F_h(h(s)) = F_{S_T}(s), \]  
(36)
\[ \text{If } h \text{ is decreasing, } F_h(h(s)) = 1 - F_{S_T}(s). \]  
(37)
Proposition 17 Assume that $U$ is concave or satisfies assumption (32), and that the marginal utility has an inverse $J$. The optimal payoff $h^*$ must satisfy:

$$h^* = J(\lambda g/\psi(F)),$$

and is characterized both by the determination of the appropriate sequence of real numbers $(s_i)_{1 \leq s_i \leq n}$ and the corresponding Lagrange parameter (see Section 4, Anticipated utility and Tversky and Kahneman cases).

Proof. The proof is based on two properties:

1. The weighting function $\psi_h$ does not depend on $h \in \mathcal{H}$.
2. Condition $h \in \mathcal{H}$, which must be taken into account for the optimization problem, does not modify the first-order condition $\frac{\partial h}{\partial h} = \lambda$, given in Proposition (8) (see Appendix B).

Corollary 18 Consider a given set $[a, b]$. If the utility function $U$ is concave on this set, $h^*$ is an increasing function of the benchmark $S_T$ if and only if the ratio $g/\psi(F)$ is a decreasing function of $S_T$. If the utility function $U$ is convex on this set, $h^*$ is an increasing function of the benchmark $S_T$ if and only if the ratio $g/\psi(F)$ is an increasing function of $S_T$. Thus, if we assume that $f$, $g$ and $\bar{\psi}$ are differentiable, then, from the optimality conditions, the optimal payoff $h^*$ is increasing according to the comparison of $\frac{\bar{\psi}(F(s))}{g/F(s)}$ with $\frac{\bar{\psi}(F(s))}{g/F(s)} f(s)$.

Proof. We deduce the result from the following properties:

- If the utility function $U$ is concave, the marginal utility $U'$ is decreasing, then $J$ is also decreasing.
- If the utility function $U$ is convex, the marginal utility $U'$ is increasing, then $J$ is also increasing.

Using the relation $\lambda = U'(h)/[g/\bar{\psi}(F)]$, standard calculus leads to:

$$h'(s) = \left(-\frac{U'(h(s))}{U''(h(s))}\right) \times \left(\frac{g'(s)}{g(s)} + \frac{\bar{\psi}'(F(s))}{\bar{\psi}(F(s))} f(s)\right).$$

(39)

According to the sign of $-U'(h(s))/U''(h(s))$ ($U$ convex/concave), $h'$ is positive if and only if we have either $\frac{g'(s)}{g(s)} \leq \frac{\bar{\psi}'(F(s))}{\bar{\psi}(F(s))} f(s)$ (if $U$ is concave) or $\frac{g'(s)}{g(s)} \geq \frac{\bar{\psi}'(F(s))}{\bar{\psi}(F(s))} f(s)$ (if $U$ is convex).

Remark 19 When there is no probability transformation, ($\psi \equiv 1$ and $\bar{\psi} \equiv 1$, standard expected utility), this condition is equivalent to the condition "$g$ is decreasing if $U$ is concave" and "$g$ is increasing if $U$ is convex".

When there exists a probability transformation, the monotonicity of the optimal payoff $h^*$ depends on the ratio of the two weighting functions, $g$ and $\bar{\psi}(F)$, associated to two fundamental modifications of the initial probability $\mathbb{P}$: for the first one, which corresponds to the asset pricing, the probability $\mathbb{P}$ "becomes"
the risk neutral probability $Q$; for the second one, which is associated to the investor’s towards risk, the probability $P$ “becomes” $\hat{P}$.

If $U$ is always concave (standard case), the optimal payoff $h^*$ is increasing if and only if the ratio $g/\hat{\psi}(F)$ is decreasing, which means that higher the risky asset value $S_T$, higher is the behavioral weighting function $\hat{\psi}(F)$ with respect to the pricing function $g$.

Introduce the function $T_0(h(s))$ defined by:

$$T_0(h(s)) = -\frac{U'(h(s))}{U''(h(s))}. \quad (40)$$

If $U$ is always concave (standard case), the function $T_0(h(s))$ is called the tolerance of risk and corresponds to the inverse of the absolute risk-aversion. As it can be seen, $h'(s)$ depends on $T_0(h(s))$. The design of the optimal payoff can also be specified. Denote

$$Y(s) = -\frac{g'(s)}{g(s)} + \frac{\hat{\psi}'(F(s))}{\hat{\psi}(F(s))} f(s). \quad (41)$$

Differentiating twice with respect to $s$, we get:

**Corollary 20** Assume that assume that $f$, $g$ and $\psi$ are twice-differentiable. Then:

$$h''(s) = [T_0'(h(s)) + \frac{Y'(s)}{Y(s)^2}] \times [T_0(h(s))Y^2(s)]. \quad (42)$$

### 3.4 The insured portfolio

This section is a generalization of Bertrand, Lesne and Prigent (2001) and Prigent (2006) to the case of rank dependent expected utility. Now, the investor introduces a specific guarantee which provides an additional insurance against risk. If for example the interest rate in non stochastic, such guarantee can be modelled by letting a function $h_0$ defined on the possible values of the benchmark $S_T$: whatever the value of $S_T$, the investor wants to get a final portfolio value above the floor $h_0(S_T)$. For instance, if $h_0$ is linear with $h_0(s) = as + b$, then, when the benchmark falls, the investor is sure of getting at least $b$ (equal to a fixed percentage of her initial investment) and if the benchmark rises, she make profits out of the rises at a percentage $a$.

The optimal payoff with insurance constraints on the terminal wealth is solution of the following problem:

$$\max_{X_T} \mathbb{E}_P[U(h(X_T))] \quad (43)$$

$$V_0 = \mathbb{E}_P[h(X_T)M_T]$$

$$h(X_T) \geq h_0(X_T)$$
As it can be seen, the initial investment $V_0$ must be higher than $\mathbb{E}_T[h_0(X_T)M_T]$ if the insurance constraint must be satisfy. We get the following result (see proof in Appendix C).

**Proposition 21** There exists an unconstrained optimal payoff $h^*$ associated to a Lagrange coefficient $\lambda_c$ such that the optimal payoff $h^{**}$ is given by:

$$h^{**} = \text{Max}(h_0, h^*) = \text{Max}(h_0, J(\lambda_c g/\hat{\psi}(F))).$$ (44)

**Remark 22** The parameter $\lambda_c$ can also be considered as a Lagrange multiplier associated to a non insured optimal portfolio but with a modified initial wealth. Indeed, when $h^{**}$ is greater than the insurance floor $h_0$, then $h^{**} = h^*$. Otherwise, $h^{**} = h_0$. Assuming that both the functions $h^*$ and $h_0$ are continuous, the payoff $h^{**}$ is a continuous function of the values of the benchmark like usually any linear combination of standard options.

### 3.5 Hedging of the optimal portfolio

As proved in Carr and Madan (1997), it is possible to explicitly identify the investment strategy that must be taken in order to achieve a given payoff $h$ that is twice differentiable. Suppose for example that the interest rate is non stochastic. The portfolio $h(S)$ is duplicated by an unique initial position of $h(S_0) - h'(S_0)S_0$ unit discount bonds, $h'(S_0)$ shares and $h(K)dK$ out-of-the-money options of all strikes $K$:

$$h(S) = [h(S_0) - h'(S_0)S_0] + h'(S_0)S + \int_{S_0}^{\infty} h''(K)(K - S)^+ dK + \int_{S_0}^{\infty} h''(K)(S - K)^+ dK. $$ (45)

Generally, $h_0$ is increasing and $h^*$ also (see corollary 8). Therefore, the optimal payoff is an increasing function of the benchmark. If $h$ is not differentiable, it is approximated by a sequence of twice differentiable payoff functions $h_n$. Then, since the payoff $h_n$ are twice differentiable, $h_n$ are duplicated by initial positions of $h_n(S_0) - h'_n(S_0)S_0$ unit discount bonds, $h'_n(S_0)$ shares and $h_n(K)dK$ out-of-the-money options of all strikes $K$:

$$h_n(S) = [h_n(S_0) - h'_n(S_0)S_0] + h'_n(S_0)S + \int_{S_0}^{\infty} h''_n(K)(K - S)^+ dK + \int_{S_0}^{\infty} h''_n(K)(S - K)^+ dK. $$

All the above properties are illustrated in next examples.
4 Examples

In what follows, previous theoretical results are illustrated for the standard expected utility, the anticipated utility and the cumulative prospect approach. Option prices are assumed to be determined in the well-known Black and Scholes framework.\(^7\)

4.1 The financial market

Suppose that the interest rate \(r\) is constant and the stock price has a Lognormal distribution given by:

\[
S_T = S_0 \exp \left[ mT + \sigma \sqrt{T} \, X \right],
\]

where the distribution of \(X\) is the standard Gaussian \(\mathcal{N}(0,1)\). For example, in a continuous-time framework, consider a geometric Brownian motion \((S_t)\), given by:

\[
S_t = S_0 \exp \left[ (\mu - 1/2\sigma^2)t + \sigma W_t \right],
\]

with \(m = (\mu - 1/2\sigma^2)\).

The probability density function (pdf) \(f\) of \(S_T\) is given by:

\[
f(s) = \frac{1}{s\sigma \sqrt{2\pi T}} \exp \left( -\frac{1}{2\sigma^2 T} \left[ \ln \left( \frac{s}{S_0} \right) - mT \right]^2 \right) \mathbb{1}_{s > 0}.
\]

The cumulative distribution function (cdf) \(F\) of \(S_T\) is given by: (\(N\) denotes the cdf of the standard normal distribution \(\mathcal{N}(0,1)\))

\[
F_S(s) = N \left[ \frac{\ln \left( \frac{s}{S_0} \right) - mT}{\sigma \sqrt{T}} \right].
\]

Introduce the following notations:

\[
\theta = \frac{\mu - r}{\sigma} \text{ (Sharpe ratio)}, \quad A = -\frac{1}{2} \theta^2 T + \frac{\theta}{\sigma} mT, \quad \chi = e^A(S_0)^{\theta}, \quad \kappa = \frac{\theta}{\sigma}.
\]

Recall that in the Black and Scholes model, since we have:

\[
W_T = \frac{\ln \left( \frac{S_T}{S_0} \right) - mT}{\sigma},
\]

we deduce that the conditional expectation \(g\) of \(\frac{\partial S_{ST}}{\partial \theta}\) under the \(\sigma\)-algebra generated by \(S_T\) is given by:

\[g(S_T) \text{ with } g(s) = \chi s^{-\kappa}.\]

\(^7\)This case is examined since it is the most used in practice. Other cases can also be considered if the Log return of the risky asset is no longer Gaussian.
We apply the previous general results to solve the optimization problem. The solutions are illustrated for the following numerical values of financial market parameters:

\[ r = 3\%, \quad \mu = 10\%, \quad \sigma = 20\%, \quad B_0 = 1, \quad S_0 = 100. \]

The initial investment is \( V_0 = 1000 \) and the time horizon \( T \) is equal to 1 (one year).

\[ \begin{align*}
\text{Cdf of the risky asset } S_T & \quad \text{Fig.2. Cdf of } S_T \text{ and Radon-Nikodym density } \\
\text{Radon-Nikodym density } dQ/dP & \quad \text{Density } g
\end{align*} \]

### 4.2 Standard expected utility

Assume that the utility function of the investor is a CRRA utility:

\[ U(v) = \frac{v^\alpha}{\alpha}, \]

with \( 0 < \alpha < 1 \) from which we deduce \( J(y) = y^{\frac{1}{1-\alpha}} \).

**Proposition 23** The optimal payoff is given by:

\[ h^*(s) = \frac{V_0 e^{rT}}{\int_0^\infty g(s) \frac{1}{\sigma^2} f(s) ds} \times g(s)^{\frac{1}{1-\alpha}}. \] (46)

Therefore, \( h^*(s) \) satisfies:

\[ h^*(s) = d \times s^\beta \quad \text{with} \quad d = c\chi^{\frac{1}{1-\alpha}}, \quad \text{and} \quad \beta = \frac{\kappa}{1-\alpha} > 0. \] (47)

**Remark 24** Note that \( h^* \) is increasing. This property is satisfied for all concave utilities, as soon as the density \( g \) is decreasing, for instance within the Black-Scholes asset pricing framework.

**Corollary 25** The concavity/convexity of the optimal payoff is determined by the comparison between the relative risk-aversion \( (1-\alpha) \) and the ratio \( \kappa = \frac{\mu - r}{\sigma^2} \), which is the Sharpe ratio divided by the volatility \( \sigma \).

- i) \( h^* \) is concave if \( \kappa < 1 - \alpha \).
- ii) \( h^* \) is linear if \( \kappa = 1 - \alpha \).
- iii) \( h^* \) is convex if \( \kappa > 1 - \alpha \).

\[ \text{See e.g. Prigent (2006, 2007).} \]
According to the financial values and the relative risk aversion, we get a convex payoff since here $\beta = 3.5 > 1$. The (approximated) corresponding position on option markets is for example $V_T = S_T + (S_T - K)^+$, which means that you buy about 21 shares of the risky asset price and 25 shares of the call with strike $K = 105$. An at-the-money option can also be used ($K = S_0 = 105$).

4.3 Behavioral case

We consider now models where the initial probabilities are transformed.

4.3.1 Anticipated utility case

Assume also that the utility function of the investor is a CRRA utility: $U(v) = \frac{v^\alpha}{\alpha}$. The functional $\varphi(f_V)$, which characterizes the probability transformation, is defined by $\varphi(f_V)(v) = f_V(v)\psi(F_V)(v)$ with:

$$\psi(F_V)(v) = w'_\gamma(F_V)(v).$$

Consider the case:

$$w'_\gamma(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}}. \tag{48}$$

We deduce:

$$w'_\gamma(p) = \frac{p^\gamma - 1 \left[(\gamma - 1)p + \gamma(1-p)^\gamma + p(1-p)^{\gamma-1}\right]}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}+1}}. \tag{48}$$

Consider also the functions $iw'_\gamma(p) = w'_\gamma(1-p)$ and $iw'_\gamma(p) = w'_\gamma(1-p)$. 

---

**Fig.3.** Optimal portfolio profile (EU case)
Fig. 4. Weighting functions and derivatives for $\gamma = 0.61$

Under the assumptions on financial parameters and on the behavioral parameter $\gamma = 0.61$, both functions $g(s) w_0^{\gamma}(F_S(s))$ and $g(s) w_0^{\gamma}(1 - F_S(s))$ have the following graphs:

Fig. 5. Ratios of the weighting transformations

The probability transformation $\hat{\Psi}$ is such that:

- $\hat{\Psi}(F_S)(s) = w_0^{\gamma}[F_S(s)]$, if $h^*$ is increasing,
- $\hat{\Psi}(F_S)(s) = w_0^{\gamma}[1 - F_S(s)]$, if $h^*$ is decreasing.

Since the utility is concave ($\alpha = 0.5$), the optimal solution $h^*(.)$ is a decreasing function of the ratios $\frac{g(.)}{w_0^{\gamma}(F_S)(.)}$ and $\frac{g(.)}{w_0^{\gamma}(1 - F_S)(.)}$. According to the variations of these functions (see Figure 5), the optimal payoff is necessarily decreasing
on the interval $[0, s^*]$ and increasing on $]s^*, \infty[$. On $[s^*, s^{**}]$, we always have $\frac{g(.)}{w_s'(F_S)(.)} \geq \frac{g(.)}{w_s'(1-F_S)(.)}$. Therefore, from the general Relation (38), the optimal payoff is given by:

$$h^*(s) = \frac{V_0 e^{r T}}{\int_0^\infty \left( \frac{g(s)}{\psi(F_S)(s)} \right) g(s) f(s) \, ds} \times \left( \frac{g(s)}{\psi(F_S)(s)} \right)^{1/\alpha}, \quad (49)$$

with

$$\hat{\psi}(F_S)(s) = w_s'[1 - F_S(s)], \text{ if } s \in [0, s^*],$$

$$\hat{\psi}(F_S)(s) = w_s'[F_S(s)], \text{ if } s \in [s^*, \infty[$$.

**Remark 26** If we assume the monotonicity of ratios $\frac{g(.)}{w_s'(F_S)(.)}$ and $\frac{g(.)}{w_s'(1-F_S)(.)}$, the determination of the optimal portfolio is much more simple. But, for this fundamental example, the shape of ratios $\frac{g(.)}{w_s'(F_S)(.)}$ and $\frac{g(.)}{w_s'(1-F_S)(.)}$ is the same as in Figure (5), whatever the choice of market and behavioral parameters (see Appendix D).

For the numerical base case with $\alpha = 0.5$ and $\gamma = 0.61$, we get the following figure:

![Fig. 6. Optimal portfolio profile (AU case)](image)

According to the financial values and the behavioral parameters, we get an optimal payoff which is first decreasing and then increasing. The (approximated) corresponding position on option markets looks like a straddle $V_T = 300 + q_c (S_T - K)^+ + q_p (K - S_T)^+$ with $q_c \approx 25$ and $q_p \approx 70$, with strike $K = 106$ (for the standard straddle, $q_c = q_p$). An at-the-money option can also be used in practice.

---

9 The two curves intersect at $s = 108$. Since $\bar{s} \notin [s^*, s^{**}]$, the optimal payoff is not continuous (jump at $s^{**}$). However, it is injective on the two intervals $[0, s^{**}]$, and $[s^{**}, \infty]$ (and piecewise differentiable). For the numerical illustration, the payoff is “almost” continuous, since $\bar{s} \approx s^{**}$.

10 $K = 106$ is approximately the minimum of the portfolio value.
Remark 27 We find almost the same type of payoff function, but with a discontinuity of this function. Indeed, with numerical values used here, we find an optimal solution of type:

\[ h = h_1\mathbb{1}_{x \leq a} + h_2\mathbb{1}_{x > a}, \]

where the control variables of the optimization problem are respectively the two payoff functions \( h_1 \) and \( h_2 \), and the bound \( a \). The first-order condition in \( a \) (if it is satisfied) leads to the continuity of the payoff function in \( a \).

Remark 28 For small risky asset values \((s \leq 80\%_0, s_0)\), the cdf \( F(s) \) is smaller than 0.44. For high risky asset values \((s \geq 120\%_0, s_0)\), the term \((1-F(s))\) is also smaller than 0.44. Therefore, \( w_i([F_S(s)] \) (resp. \( w_i(1-F_S(s)) \) are higher than \( F_S(s) \) and (resp. \( 1-F_S(s) \)). This overweighing of relatively “rare” events leads the investor to adopt an option positioning based on realizations of relatively extreme events (significant drops or rises of the underlying asset). This strategy usually corresponds to an anticipation of a high volatility. However, as seen in Figure 3, for an investor who maximizes the standard expected utility, such anticipation leads to a more convex payoff but it must be always increasing since the risk-neutral density \( q \) is decreasing and the inverse \( J \) of the marginal utility is decreasing. This means that this investor wants a high payoff only for market rises, and not when the financial market drops significantly.

4.3.2 Kahneman and Tversky case

According to Tversky and Kahneman (1992) who introduce the cumulative prospect theory (CPT), the investor has not the same kind of utility \( U \) when she suffers from losses (\( U \) convex) and when she benefits from gains (\( U \) concave). To illustrate how the general result given in Proposition (17) can be applied to the Tversky and Kahneman case, assume that the utility function of the investor is defined by:

\[
U(v) = a(v-v^*) + \frac{(v^*-v)^{\alpha_1}}{\alpha_1}, \quad \text{for } v \leq v^*,
\]

\[
U(v) = a(v-v^*) + \frac{(v^*-v)^{\alpha_2}}{\alpha_2}, \quad \text{for } v > v^*.
\]

with \( 1 < \alpha_1, 0 < \alpha_2 < 1 \) and \( a = (v^*)^{(\alpha_1-1)/15}. \)

We deduce:

\[
J(y) = v^* - (a-y)^{1/\alpha_1}, \quad \text{for } y \leq a, \\
J(y) = v^* + (y-a)^{1/\alpha_2}, \quad \text{for } y > a.
\]

Consider the following behavioral parameters: \( \alpha_1 = 1.5, \alpha_2 = 0.5, \) and \( w_{\gamma^-} = 0.69, w_{\gamma^+} = 0.61. \)

The functional \( \varphi(f_V) \) is defined by \( \varphi(f_V)(v) = f_V(v)\psi(F_V)(v) \) with

\[
\psi(F_V)(v) = \frac{w_{\gamma^-}[F(v)]1_{v \leq v^*} + w_{\gamma^+}[1-F(v)]1_{v \geq v^*}}{w_{\gamma^-}[F(v^*)] + w_{\gamma^+}[1-F(v^*)]}. 
\]
and

\[ w_\gamma(p) = \frac{p^\gamma}{(p^\gamma + (1-p)\gamma)} \]

In what follows, we assume that the reference level \( v^* \) is equal to the initial portfolio value \( V_0 \) (loss for \( V_T < V_0 \) and gain for \( V_T > V_0 \)). We have to determine the optimal portfolio profile \( h^* \) such that the portfolio value \( V_T \) is equal to the payoff \( h^*(S_T) \).

We have:

\[ \mathbb{E}_\theta[V_T] = \int U[v] f_V(v) \psi(F_V(v)) dv. \]

For a given subset \([a, b]\) on which \( h^*(\cdot) \) is injective and increasing, we have

\[ \int_{[h^*]^{-1}(a)}^{[h^*]^{-1}(b)} U[v] f_V(v) \psi(F_V(v)) dv = \int_a^b U[h(s)] f_S(s) w'_1(F_S(s)) ds \]

For a given subset \([a, b]\) on which \( h^*(\cdot) \) is injective and decreasing, we have:

\[ \int_{[h^*]^{-1}(a)}^{[h^*]^{-1}(b)} U[v] f_V(v) \psi(F_V(v)) dv = \int_a^b U[h(s)] f_S(s) w'_2(1 - F_S(s)) ds \]

Therefore, the probability transformation \( \hat{\Psi} \) is such that: for a given subset \([a, b] \)

\[ \begin{align*}
\text{If } h^*(s) &< V_0 \text{ for } s \in ]a, b[ \text{ ("Loss")}: \\
\hat{\psi}(F_S(s)) &= w'_\gamma[F_S(s)], \text{ if } h^* \text{ is increasing on } ]a, b[, \\
&= w'_{\gamma^-}[1 - F_S(s)], \text{ if } h^* \text{ is decreasing on } ]a, b[,
\end{align*} \]

\[ \begin{align*}
\text{If } h^*(s) &> V_0 \text{ for } s \in ]a, b[ \text{ ("Gain")}: \\
\hat{\psi}(F_S(s)) &= w'_\gamma[1 - F_S(s)], \text{ if } h^* \text{ is increasing on } ]a, b[, \\
&= w'_\gamma[F_S(s)], \text{ if } h^* \text{ is decreasing on } ]a, b[,
\end{align*} \]

where the probability transformation \( \hat{\Psi} \) is defined on each \([a, b]\) as in Relation (52).

To determine precisely the payoff \( h^* \), we have to determine the increasing sequence \( (s_i)_{1 \leq i \leq n} \) such that \( \forall i \), the payoff \( h^* \) is injective on \([s_i, s_{i+1}]. \)

Deriving the Lagrangian function with respect to the real numbers \( s_i \), their values correspond to the solutions of equations defined as follows:

\[ \text{Find } x \text{ such that } w'_\gamma[1 - F_S(s_i)] = w'_{\gamma_2}[F_S(s_i)] \text{ with } \gamma_1, \gamma_2 \in \{\gamma^-, \gamma^+\}. \]

This subdivision is assumed to be “maximal” in the following sense: if \( h^* \) is injective on \([\gamma, \gamma_+],[s_i, s_{i+1}] \) with \([s_i, s_{i+1}] \subset [\gamma, \gamma_+], \), then \( s_i = \gamma \) and \( s_{i+1} = \gamma_+ \).
The optimal payoff is given by:

\[ h^*(s) = V_0 - \left( \lambda \frac{g(s)}{\psi(F_S)(s)} \right)^{-\frac{1}{\alpha_1 - 1}}, \text{ for } h^*(s) \leq V_0, \] (53)

\[ h^*(s) = V_0 + \left( \lambda \frac{g(s)}{\psi(F_S)(s)} \right)^{-\frac{1}{\alpha_2 - 1}}, \text{ for } h^*(s) > V_0, \] (54)

From previous relations, the optimal solution \( h^*(\cdot) \) is an increasing or decreasing function according to the monotonicities of the four ratios \( \frac{g(\cdot)}{w_{x^-}(1-F_S)(\cdot)} \), \( \frac{g(\cdot)}{w_{x^+}(F_S)(\cdot)} \) and \( \frac{g(\cdot)}{w_{x^+}(1-F_S)(\cdot)} \). The variations of these functions look like those in previous Figure 5.

Without weighting transformation, we get: (maximal loss \( \simeq -25\% \))

![Graph 1](image1)

**Fig 7. Optimal profiles for the KT case**

Using previous general result (44) and the same assumptions on financial and behavioral parameters, we get the following figure for an insurance function \( h_0(S_T) = 90\%.V_0 + 5\%.S_T \).

We can also try to determine the shape of the static optimal payment for an investor having a behavior towards the risk as described by the cumulative prospect theory.

![Graph 2](image2)

**Fig 8. Optimal insured profile for the AT case**
5 Conclusion

Using the rank dependent expected utility theory, the optimal payoff on the horizon wealth have been determined for a large class of models. This kind of results prove that derivative assets have to be introduced in the portfolio to maximize the expected utility of investors (with and without rank dependence). The optimal solution clearly depends on the risk aversion of the investor and, under insurance constraints, it also depends on the kind of guarantee at maturity. The optimal portfolio is determined for quite general utility functions, stock prices and insurance constraints. In the no guarantee case, the concavity/convexity of the portfolio profile is determined from the behavioral parameters (degree of risk aversion and probability transformation) and from the financial market performance, for example a Sharpe type ratio. This kind of result still holds according to the insurance constraint at maturity. The same kind of results would be extended in the continuous-time, using for example the dynamic completeness. Assuming that it corresponds to the conditional expectation of the density of the transformed probability with respect to the initial one, the result is rather straightforward (see for example Cox and Huang (1989), Cvitanic and Karatzas (1996)) and, with insurance constraints, El Karoui et al. (2005) and Prigent (2006). However, the theory of the dynamic probability transformation is not yet achieved and such hypothesis is not clearly justified while, for the static case, the information corresponds only to the observation at initial time.

References


Appendix

A. Subdifferentiability and indicator function

In what follows, we refer to Ekeland and Turnbull (1983) for main definitions and properties of subdifferentials.

The indicator function of a subset $K$ is denoted by $\delta_K$ and defined by:

$$\delta_K(h) = \begin{cases} 0 & \text{if } h \in K, \\ +\infty & \text{if } h \notin K. \end{cases}$$

If $K$ is closed and convex, $\delta_K$ is lower semi-continuous and convex.

Recall the notion of subdifferentiability:

Let $\mathcal{V}$ denote a Banach space and $<..,>$ the duality symbol.

1) For any function $G$ defined on $\mathcal{V}$ with values in $\mathbb{R} \cup \{+\infty\}$, a continuous affine functional $L: \mathcal{V} \to \mathbb{R}$ everywhere less than $G$ (i.e. $\forall v \in \mathcal{V}, L(v) \leq G(v)$) is exact at $v^*$ if $L(v^*) = G(v^*)$.

2) A function $G: \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ is subdifferentiable at $v^*$ if there exists a continuous affine functional $L(.) = <..,v> - a$, everywhere less than $G$, which is exact at $v^*$. The slope $v_0$ of such an $L$ is a subgradient of $G$ at $v^*$. The set of all subgradients of $G$ at $v^*$ is the subdifferential of $G$ at $v^*$ and is denoted by $\partial G(v^*)$.

Recall the following characterization:

$v_c \in \partial G(v^*)$ iff $G(v^*) < +\infty$ and $\forall v \in \mathcal{V}, <v - v^*, v_c > + G(v^*) \leq G(v)$. (55)

B. Determination of optimal piecewise monotone profile

We search the solution on a given subset $\mathcal{H}$ of $L^2(\mathbb{R}^+, \mathbb{P}_{S_T})$ which is defined as follows: there exists an increasing sequence $(s_i)_{1 \leq i \leq n}$ such that $\forall i$, the payoff $h^*$ is monotone on $[s_i, s_{i+1}]$. The monotony on each $[s_i, s_{i+1}]$ is fixed. Thus, both $\bar{\psi}_h$ and $\hat{\psi}_h^+$ do not depend on $h$.

In that case, any profile $h$ corresponds to the choice of the $n$ functions $h_i$ which are the restrictions of $h$ on each interval $[s_i, s_{i+1}]$.

These functions belong respectively to $L^2([s_i, s_{i+1}], \mathbb{P}_{S_T})$. Consider the subset $K_i$ of $L^2([s_i, s_{i+1}], \mathbb{P}_{S_T})$ corresponding to either all the increasing functions $h_i$ of $L^2([s_i, s_{i+1}], \mathbb{P}_{S_T})$, either all the decreasing ones, according to the choice of the monotony on $[s_i, s_{i+1}]$. The subset $K_i$ is a cone which is convex and closed.

Lemma 29 The subdifferential of any $h^*_i$ in $K_i$ only contains the null function.

Proof. From characterization (55), any subgradient $h_i$ of $\partial \delta_{K_i}(h^*_i)$ must satisfy:

$$\forall h \in L^2([s_i, s_{i+1}], \mathbb{P}_{S_T}), <h - h^*, h_i > \geq \partial \delta_{K_i}(h). \quad (56)$$
Therefore, for any \( h \) in \( K \),
\[
\langle h - h^\star, h \rangle = \int_{s_i}^{s_{i+1}} (h - h^\star)(s) h_t(s) \mathbb{P}_{S_t}(ds) \leq 0.
\]

Consider now any subinterval \([a_i, b_i]\) of \([s_i, s_{i+1}]\). For any number \( c \), set \( h_c = h^\star + c \) on \([a_i, b_i]\) and \( h_c = h^\star \) on \([s_i, s_{i+1} - [a_i, b_i]\). Then, for any number \( c \),
\[
\int_{s_i}^{s_{i+1}} (h - h^\star)(s) h_t(s) \mathbb{P}_{S_t}(ds) = c \int_{a_i}^{b_i} h_t(s) \mathbb{P}_{S_t}(ds) \leq 0.
\]
Therefore, for any \([a_i, b_i]\) \( \subset [s_i, s_{i+1}]\) \( \mathbb{R}^3 \),
\[
\int_{s_i}^{s_{i+1}} (h - h^\star)(s) h_t(s) \mathbb{P}_{S_t}(ds) = 0. \]
Consequently, we deduce that \( h_t = 0 \) (a.s.).

Using Proposition (9) and Lemma (29), we deduce:
\[
\partial (\hat{\Gamma}_U + \delta K) = \partial (\hat{\Gamma}_U).
\]

C. Determination of the optimal insured portfolio

To solve the optimization problem with insurance constraints, introduce the sets
\[
K_1 = \{ h \in L^2(\mathbb{R}^3, \mathbb{P}_{X_T}) | V_0 = \mathbb{E}_\mathcal{F}[h(X_T)M_T] \}
\]
and
\[
K_2 = \{ h \in L^2(\mathbb{R}^3, \mathbb{P}_{X_T}) | h \geq h_0 \}.
\]
The set \( K = K_1 \cap K_2 \) is a convex set of \( L^2(\mathbb{R}^3, \mathbb{P}_{X_T}) \).
Denote by \( \partial \delta K \) the subdifferential of \( \delta K \). The optimization problem is equivalent to:
\[
Max_h \left( \mathbb{E}[(h(X_T)\hat{\psi}[F(X_T)]) - \delta K(h)] \right) \quad (57)
\]
The optimality conditions leads to:

**Lemma 30** There exists a scalar \( \lambda_c \) and a function \( h_c \) defined on \( L^2(\mathbb{R}^3, \mathbb{P}_{X_T}) \) such that:
\[
h^\star = J(\lambda_c g / \hat{\psi}[F] + h_c),
\]
where \( \lambda_c \) is solution of:
\[
V_0 = \int_0^\infty J[yg / \hat{\psi}[F] + h_c(x)] g(x) f(x) dx.
\]
and \( h_c \in \partial \delta K_2(h^\star) \).

**Proof.** Consider the Nemitsky functional \( \Gamma_U(X_T) = \mathbb{E}[U(h(X_T))\hat{\psi}[F(X_T)]] \). Denote by \( G \) the functional defined by
\[
G(h) = (-\Gamma_U(X_T) + \delta K(h)).
\]
The optimization problem

\[ \text{Max}_h \left( \mathbb{E}[U(h(X_T))] - \delta_K(h) \right) \] (58)

is equivalent to the following:

\[ -\text{Min}_h \left( G(h) \right), \] (59)

where \( G \) is subdifferentiable and \( h^{**} \) is solution of the optimal problem iff \( 0 \in \partial G(h^{**}) \).

\( \Gamma_U \) is continuous and, due to the kind of constraints, \( \delta_K \) is continuous. Thus:

\[ \partial G = \partial(-\Gamma_U + \delta_K) = \partial(-\Gamma_U) + \partial(\delta_K). \]

Moreover, since \( \Gamma_U \) is differentiable then \( \partial(\Gamma_U) = \{ \Gamma_U^' \} \). Additionally, we have:

\[ \partial(\delta_K) = \partial(\delta_{K_1}) + \partial(\delta_{K_2}). \]

From the characterization (55), \( h_1 \in \partial \delta_{K_1}(h^{**}) \) if and only if:

\[ \forall h, \int_0^\infty (h - h^{**})(x)h_1(x)f(x)\tilde{\psi} [F(x)] dx + \delta_{K_1}(h^{**}) \leq \delta_{K_1}(h). \]

In particular, \( \forall h \in K_1, \int_0^\infty (h - h^{**})(x)h_1(x)f(x)\tilde{\psi} [F(x)] dx \leq 0 \), from which we deduce that \( h_1 \) is orthogonal to any orthogonal function to the subspace generated by \( g \). Thus, there exists a scalar \( \lambda_c \) such that \( h_1 = \lambda_c g \).

To conclude, property \( 0 \in \partial L(h^{**}) \) holds if and only if there exists \( h_c \in \partial \delta_{K_2}(h^{**}) \) such that:

\[ 0 = -U'(h^{**}) + \lambda_c g + h_c. \]

To explain more precisely the condition \( h_c \in \partial \delta_{K_2}(h^{**}) \), assume that the insurance constraint \( h_0 \) and \( h_c \) are continuous.\(^{12}\) Consequently, the optimal payoff \( h^{**} \) is continuous.

**Lemma 31** Under the above assumption, the function \( h_c \) satisfies the following property:

1) If on an product \( I \) of intervals of values of \( X_T, h^{**}(X_T) > h_0(X_T) \) then \( h_c \) is equal to 0 on \( I \).

2) If on an product \( I \) of intervals of values of \( X_T, h^{**}(X_T) = h_0(X_T) \) then \( h_c \) is negative on \( I \).

**Proof.** Recall that from the characterization (55), we deduce:

\[ \forall h \in L^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T}), \int_0^\infty (h - h^{**})(x)(h_c)(x)f(x)\tilde{\psi} [F(x)] dx + \delta_{K_2}(h^{**}) \leq \delta_{K_2}(h). \]

\(^{12}\)Such properties are always verified in practice.
In particular:

\[ \forall h \in L^2(\mathbb{R}^3, \mathbb{P}_{X_T}) \text{ such that } h \geq h_0, \int_0^\infty (h-h^{**})(x)h_c(x)f(x)\widehat{\psi}[F(x)] \, dx \leq 0. \]

From previous condition, we deduce:

If \( \forall x \in I, h^{**}(x) > h_0(x) \) then, on each compact subinterval \( I_{co} \) of \( I \), we can consider \( m_{co} \) equal to the minimum of \( h^{**} - h_0 \) on \( I_{co} \). The parameter \( m_{co} \) is non negative. Now, consider the function \( h \) equal to \( h^{**} - (m_{co}1_{I_{co}}) \). By construction, \( h > h_0 \) everywhere. Therefore, the relation

\[ \int (h-h^{**})(x)h_c(x)f(x)\widehat{\psi}[F(x)] \, dx \leq 0 \]

is true. This property implies:

\[ m_{co} \int_{I_{co}} h_c(x)f(x)\widehat{\psi}[F(x)] \, dx \geq 0. \]

On the other hand, by letting \( h \) equal to \( h^{**} + a \) where \( a \) is a non negative constant, \( a \int_{I_{co}} h_c(x)f(x)\widehat{\psi}[F(x)] \, dx \leq 0 \). Consequently, since for all \( I_{co} \),

\[ \int_{I_{co}} h_c(x)f(x)\widehat{\psi}[F(x)] \, dx = 0, \]

\( h_c \) is equal to 0 on the product of intervals \( I \). To prove (2), consider for all \( I_{co} \), a function \( h \) equal to \( h^{**} + a1_{I_{co}} \). Then, the relation \( \int (h-h^{**})(x)h_c(x)f(x)\widehat{\psi}[F(x)] \, dx \leq 0 \) is a \( \int_{I_{co}} h_c(x)f(x)\widehat{\psi}[F(x)] \, dx \leq 0 \), from which the negativity of \( h_c \) on \( I \) is deduced.

Since the general result in previous lemma is an extension of the Kuhn-Tucker theorem to infinite dimension, it is well-known that the determination of \( h^{**} \) implies to compare all possible solutions of the kind (44). However, this problem can be solved if the payoff must be continuous.

**Lemma 32** Under the previous assumptions on the utility \( U \), there is one and only one continuous optimal payoff, associated to the unique solution \( \lambda_c \) of the budget equation.

**Proof.** From assumptions on the marginal utility \( U' \), we deduce that its inverse \( J \) is a continuous and decreasing function with:

\[ \lim_{s \to +} J = +\infty \text{ and } \lim_{s \to -\infty} J = 0. \]

Thus, for all \( s \), the function \( \lambda \to h^{**}(\lambda, x) = \max(h_0(x), \tilde{h}(\lambda, x)) \) is continuous and decreasing. Therefore, the function \( \lambda \to \mathbb{E}_Q[h^{**}(\lambda, X_T)] \) is continuous and decreasing from \( +\infty \) to \( \mathbb{E}_Q[h_0(\lambda, X_T)] \) which is lower than the initial investment \( V_0 \). From the intermediate values theorem and by monotonicity, the result is deduced. ■
D. Study of the monotonicity of ratios $\frac{g(s)}{w'_0(F_S(s))}$ and $\frac{g(s)}{w'_0(1-F_S(s))}$

We have:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp \left[ -\frac{x^2}{2} \right] dx < \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} |x| \exp \left[ -\frac{x^2}{2} \right] dx. $$

Since

$$\int_{-\infty}^{x} |x| \exp \left[ -\frac{x^2}{2} \right] dx = \int_{x}^{\infty} x \exp \left[ -\frac{x^2}{2} \right] dx = \exp \left[ -\frac{x^2}{2} \right],$$

we obtain the following inequality:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp \left[ \frac{x^2}{2} \right] dx < \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{x^2}{2} \right].$$

Then, we deduce the following result.

**Lemma 33** (Study of $N(\ln(s))$ when $s \to 0^+$.)

The distribution function $N$ of the standard Gaussian distribution satisfies:

$$N(\ln(s)) < \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{\ln(s)^2}{2} \right],$$

which implies:

$$s^{-a} \cdot [N(\ln(s))]^b \to 0, \forall a, b > 0.$$

**Proof.** We have :

$$s^{-a} \cdot [N(\ln(s))]^b < \left( \frac{1}{\sqrt{2\pi}} \right)^b \exp \left[ -\frac{b \ln(s)^2}{2} \right] \exp \left[ -a \ln(s) \right]$$

$$< \left( \frac{1}{\sqrt{2\pi}} \right)^b \exp \left[ -\frac{1}{2} \left(b \ln(s)^2 + 2a \ln(s) \right) \right].$$

As we have : $\ln(s) \to -\infty$, we deduce the result. $\blacksquare$

By applying previous lemma, we deduce:

$$\lim_{s \to 0^+} \frac{g(s)}{w'_0(F_S(s))} = 0 \quad \text{and} \quad \lim_{s \to 0^+} \frac{g(s)}{w'_0(1-F_S(s))} = 0.$$

Recall that the derivative of the weighting function is given by:

$$w'_0(p) = \frac{p^{\gamma-1} \left[ (\gamma - 1)p + \gamma (1-p) \gamma + p(1-p)^{\gamma-1} \right]}{(p^{\gamma} + (1-p)^{\gamma})^{\frac{\gamma+1}{\gamma+1}}}.$$

Therefore, it is equivalent to $p^{\gamma-1}$ near 0. Consequently, the study of the limit of two ratios leads exactly to use the result of Lemma (33).