Global Stability of Unique Nash Equilibrium in Cournot Oligopoly and Rent-Seeking Game

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February 3, 2007

Abstract
A sufficient condition is derived for the global stability of unique Nash equilibrium in aggregative game. The condition is applied to investigate the global stability of Nash equilibrium in rent-seeking game and that of Nash-Cournot equilibrium in Cournot oligopoly without product differentiation.

JEL Classification Numbers: C72, D43, L13.
Key Words: rent-seeking, Nash equilibrium, global stability, Cournot oligopoly

* We would like to thank Ferenc Szidarovszky for invaluable comments and suggestions on an earlier version of this paper. All remaining errors are of course ours.
1. Introduction

In spite of abundant literatures on the stability of the Cournot oligopoly equilibrium since the seminal work of Theocharis (1960), only restrictive conditions have been found for the global stability. Hahn (1962) has derived a set of general stability conditions on the basis of a continuous output adjustment system in which a firm’s rate of change of actual output is proportional to the difference between its profit-maximizing and actual outputs. Okuguchi (1964) has extended his result using a more general adjustment system. Al-Nowaihi and Levine (1985) have presented a counter-example to the Hahn-Okuguchi result. Okuguchi (1976), on the other hand, has given a proof of the global stability which is exempt from the criticism. Al-Nowaihi and Levine (1985) have derived a set of conditions which is sufficient for the global stability of the Nash-Cournot equilibrium. Their conditions imply that all firms’ reaction functions are upward sloping, that is, the game is submodular.¹ According to them the equilibrium is globally stable if the number of firms less than or equal to 5. Corchón (2001) has proved the global stability of the equilibrium adopting Hahn’s method of proof as well as taking into account the property of aggregative game for Cournot oligopoly without product differentiation. His proof, however, is not free from the same defect as in Hahn (1962) and Okuguchi (1964).

It is well known that the Cournot oligopoly without product differentiation (hereafter, Cournot game) is a submodular aggregative game under usual or traditional assumptions.² Most works mentioned above have studied the global stability of the

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¹ In other words, firms’ actions are strategic substitutes in the terminology of Bulow et al. (1983).
² If the inverse demand function is linear and each firm’s cost function is convex, the Cournot oligopoly without product differentiation is a submodular aggregative game.
Nash-Cournot equilibrium mainly in the submodular Cournot game. However, the Cournot game can be supermodular. Recently, some researchers including Vives (1990) and Amir (1996) have analyzed the supermodular Cournot game. Vives (1999, Theorem 2.11) implies that the unique Nash equilibrium in the supermodular Cournot game is globally stable under Hahn’s best reply dynamics. As for the traditional submodular Cournot game, as already mentioned, Al-Nowaihi and Levine (1985) have proved that the Nash-Cournot equilibrium is globally stable if the number of firms is less than or equal to 5. However, the Cournot game may not be either supermodular or submodular. That is, it is possible that some firms have downward sloping reaction functions and at the same time other firms have upward sloping reaction functions in the Cournot game. It is also possible that a Cournot firm has a non-monotone reaction function. This paper examines the global stability of the equilibrium in the general Cournot game which allows such cases. To do so, this paper derives a set of conditions which ensures that a unique (interior) Nash equilibrium in a general aggregative game is globally stable, and then applies this general result to the Cournot game and a rent-seeking game. The rent-seeking game is an aggregative game where players’ reaction functions are non-monotone under the standard assumptions in the literature of rent-seeking.

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3 In the terminology of Bulow et al. (1983), the game can have strategic complementarities.
4 As for local stability, Dastidar (2000, Proposition 3) derives conditions for the Nash equilibrium to be locally stable in the Cournot game which has firms with downward sloping reaction functions and firms with upward sloping reaction functions.
2. Aggregative Game

Consider a general aggregative game with \( n \) players.\(^5\) Player \( i \)'s payoff function is given as \( U_i(x_i, X) \), where a non-negative number \( x_i \) is player \( i \)'s choice of strategy and

\[
X = \sum_{j=1}^{n} x_j.
\]

Define

\[
h'(x_i, X) \equiv \frac{\partial}{\partial x_i} U_i(x_i, X) + \frac{\partial}{\partial X} U_i(x_i, X).
\]

At an interior Nash equilibrium \( x^* = (x_1^*, x_2^*, \ldots, x_n^*) \), \( h'(x_i^*, X^*) = 0 \) for all \( i \), where \( X^* = \sum_{j=1}^{n} x_j^* \). Assume that \( x_i \) as a function of continuous time \( t \) is adjusted according to

\[
\frac{d}{dt} x_i = k_i h'(x_i, X),
\]

where a positive number \( k_i \) denotes speed of adjustment. Define partial derivatives of \( h' \) as \( h'_i \equiv \frac{\partial}{\partial x_i} h'(x_i, X) \) and \( h'_i \equiv \frac{\partial}{\partial X} h'(x_i, X) \). We then have the following stability theorem.

**Theorem 1:** The unique Nash equilibrium in the aggregative game is globally stable if

\[
h'_1 + h'_2 < 0,
\]

\[
2k_i |h'_1 + h'_2| > \sum_{j=1, j \neq i}^{n} |k_j h'_1 + k_i h'_2|
\]

and

\(^5\) See Corchón (2001) and Okuguchi (1993) for more details on the aggregative game.
\[
\lim_{x_i \to -\infty} h'(x_i, x_i + X_{-i}) = -\infty
\]  

(4)

are satisfied for all feasible strategies and for all \(i\) and \(j \neq i\), where \(X_{-i} = \sum_{j \neq i} x_j\).

**Proof:** The following proof assumes the uniqueness of the equilibrium and all mathematical regularities required for applying the Lyapunov method. Define a Lyapunov function by

\[
V(x) = \frac{1}{2} \sum_{i=1}^{n} h'(x_i, X)^2,
\]

(5)

where \(x = (x_1, x_2, \ldots, x_n)\). Since \(x_i\) is non-negative for all \(i\), \(\|x\| = \sum_{i=1}^{n} x_i^2\) is infinite if and only if \(x_i\) is infinite for some \(i\). Hence, condition (4) implies that \(V(x) \to \infty\) as \(\|x\| \to \infty\). Differentiation of (5) with respect to \(t\) yields

\[
\frac{dV}{dt} = \mathbf{hJh}' = \frac{1}{2} \mathbf{h(J + J')} \mathbf{h}',
\]

(6)

where \(\mathbf{h} = (h', h^2, \ldots, h^n)\) and

\[
\mathbf{J} = \begin{bmatrix}
  k_1 (h_1^2 + h_1^2) & k_2 h_1^2 & \cdots & k_n h_1^2 \\
  k_1 h_2^2 & k_2 (h_1^2 + h_2^2) & \cdots & k_n h_2^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  k_1 h_n^2 & k_2 h_n^2 & \cdots & k_n (h_1^n + h_2^n)
\end{bmatrix}.
\]

(7)

If the symmetric matrix \(\mathbf{J} + \mathbf{J}'\) in (6) is negative definite, the unique Nash equilibrium is globally stable. A sufficient condition for \(\mathbf{J} + \mathbf{J}'\) to be negative definite is conditions (2) and (3), which ensure that \(\mathbf{J} + \mathbf{J}'\) is negative dominant diagonal.\(^6\)

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\(^6\) See McKenzie (1960).
3. Applications

3.1 Cournot Oligopoly without Product Differentiation

In Cournot oligopoly without product differentiation,

\[ U_i(x_i, X) = x_i P(X) - C_i(x_i), \quad (8) \]

\[ h_i'(x_i, X) = P(X) + x_i P'(X) - C'_i(x_i), \quad (9) \]

where \( x_i \) is firm \( i \)'s output, \( P(X) \) with \( P'(X) < 0 \) is the inverse demand function and \( C_i \) is firm \( i \)'s cost function. Al-Nowaihi and Levine (1985, Theorem 6) prove that if

\[ h'_i(x_i, X) = P'(X) - C''_i(x_i) < 0 \quad \text{and} \quad h''_i(x_i, X) = P''(X) + x_i P''(X) < 0 \quad \text{for all } \ i, \]

which implies that all firm’s reaction function are downward sloping, then the unique Nash-Cournot equilibrium is globally stable for \( n \leq 5 \).

Assume

\[ h'_i(x_i, X) + h''_i(x_i, X) = 2P'(X) + x_i P''(X) - C''_i < 0 \quad (10) \]

for condition (2) of Theorem 1 to hold. If

\[ \lim_{x_i \to \infty} C'_i(x_i) = \infty \quad (11) \]

for all \( i \), condition (4) is satisfied. If \( k_i = k \) for all \( i \), condition (3) can be simplified as

\[ (3-n)(2P' + x_i P'') - 2C''_i < X_i P''. \quad (12) \]

Hence, Theorem 1 proves

**Theorem 2:** If \( k_i = k \) for all \( i \) and assumptions (10)-(12) are satisfied, the unique Nash-Cournot equilibrium is globally stable.
A few words on the theorem are in order. If the inverse demand function is convex and satisfies

\[ 2P'(X) + x_i P''(X) \leq 0, \]  

(13)

firm \( i \)'s marginal cost (strictly) increases and approaches to infinity as \( x_i \) increases, and the number of firms is less than or equal to 3, then the unique Nash-Cournot equilibrium is globally stable. Inequality (12) may hold even if \( n > 3 \), provided that the rate of change of the marginal cost is sufficiently large. More importantly, \( h_i(x_i, X) = P'(X) + x_i P''(X) \) can be positive under assumption (13), that is, some firms can have upward sloping reaction under assumption (13). Remember that such a case is excluded in Al-Nowaihi and Levine (1985, Theorem 6) who prove that if the number of firms is less than or equal to 5, the unique Nash equilibrium is globally stable in the traditional submodular Cournot game.\(^7\)

3.2 Rent Seeking Game

Rent-seeking game first formulated by Tullock (1980) has been much discussed and extended by many economists. Perez-Castrillo and Verdier (1992) have analyzed the existence of Nash equilibrium in a rent-seeking game with more than two agents where lottery production functions are characterized by the logit functions as in Tullock. Szidarovszky and Okuguchi (1997) have proven the existence of a unique Nash equilibrium in a rent-seeking game, where each agent has a lottery production function which is

\(^7\) As for local stability, Dastidar (2000, Proposition 3) proves that if only one firms has upward sloping reaction function and other firms have downward sloping reaction function, the unique Nash-Cournot equilibrium is locally stable under the continuous adjustment process (1) and that the more the number of firms with upward sloping reaction function, the more severe his local stability condition.
concave in his expenditure for rent-seeking activity. Chiarella and Szidarovszky (2002) have analyzed asymptotic behavior of a dynamic rent-seeking game and have derived several sufficient conditions for the local stability or instability of the Nash equilibrium. In this section we will analyze the global stability of the Nash equilibrium in a rent-seeking game.

We apply Theorem 1 to derive the global stability condition for the Nash equilibrium in a general rent-seeking game as formulated by Szidarovszky and Okuguchi (1997). Let \( n \) be the number of rent-seeking agents. The rent these agents seek is normalized to be one. Let \( y_i \) be agent \( i \)'s expenditure on his rent-seeking activity. All agents are assumed to be risk neutral. The probability of winning the rent is assumed to be

\[
p_i = \frac{f_i(y_i)}{\sum_{j=1}^{n} f_j(y_j)},
\]

where \( f_i(y_i) \) can be interpreted as agent \( i \)'s production function for lotteries. Following Szidarovszky and Okuguchi (1997), assume that \( f_i \) satisfies

\[
f_i(0) = 0, \quad f_i' > 0 \quad \text{and} \quad f_i'' < 0 \quad \text{for all} \quad i.
\]  

(14)

In addition, assume

\[
\lim_{y_i \to \infty} f_i' = 0 \quad \text{for all} \quad i.
\]  

(15)

If \( f_i(y_i) = \alpha_i y_i^r \) with \( \alpha_i > 0 \) and \( r \in (0,1) \), assumptions (14) and (15) are satisfied.

Agent \( i \)'s expected utility is

\[
u_i(y) = \frac{f_i(y_i)}{\sum_{j=1}^{n} f_j(y_j)} y_i,
\]

(16)
where \( \mathbf{y} = (y_1, y_2, \ldots, y_n) \). If all \( y_i = 0 \), then \( u_i \) is defined to be zero. Agent \( i \) is assumed to maximize his or her expected utility (16) with respect to \( y_i \). Define \( x_i = f_i(y_i) \) to transform the expected utility of agent \( i \) into a function of \( x_i \) and \( X = \sum_{j=1}^{n} x_j \):

\[
U_i(x_i, X) = \frac{x_i}{X} - g_i(x_i),
\]

where \( g_i = f_i^{-1} \). Agent \( i \)'s original maximization problem is equivalent to the one of maximizing the expected utility (16') with respect to \( x_i \). Note that this transformation has made the rent-seeking contest an aggregative game. Note also that assumption (14) on \( f_i \) implies

\[
g_i(0) = 0, \quad g_i' > 0 \text{ and } g_i'' > 0 \text{ for all } i
\]

and assumption (15) implies

\[
\lim_{x_i \to \infty} g_i' = \infty \text{ for all } i.
\]

From (16'), it is easy to get

\[
h_i'(x_i, X) = \frac{\partial}{\partial x_i} U_i(x_i, X) + \frac{\partial}{\partial X} U_i(x_i, X) = \frac{1}{X} - g_i'(x_i) - \frac{x_i}{X^2},
\]

\[
h_i'(x_i, X) = \frac{\partial}{\partial x_i} h_i'(x_i, X) = -g_i''(x_i) - \frac{1}{X^2},
\]

\[
h_i''(x_i, X) = \frac{\partial}{\partial X} h_i'(x_i, X) = -\frac{1}{X^2} + \frac{2x_i}{X^3}.
\]

Now we are ready to state
Theorem 3: If $k_i = k$ for all $i$ and assumptions (14) and (15) are satisfied, the unique Nash equilibrium of the rent-seeking game is globally stable if

$$-X^3 g_i''(x_i) < (n-4)(x_i - X)$$

is satisfied.

Proof: Assumption (15) implies condition (4) of Theorem 1. Since

$$J + J' = \frac{k}{X^3} \begin{bmatrix}
-2X_i - X^3 g_i''(x_i) & (x_i + x_j) - X & \cdots & (x_i + x_n) - X \\
(x_2 + x_1) - X & -2X_2 - X^3 g_2''(x_2) & \cdots & (x_2 + x_n) - X \\
\vdots & \vdots & \ddots & \vdots \\
(x_n + x_1) - X & (x_n + x_2) - X & \cdots & -2X_n - X^3 g_n''(x_n)
\end{bmatrix},$$

condition (2) of Theorem 1 is satisfied under assumption (14). Furthermore, since $(x_i + x_j) - X \leq 0, i \neq j$, condition (3) of Theorem 1 is equivalent to condition (17).

As already mentioned, all player’s reaction functions are not monotone in the rent seeking game, since $h_i'(x_i, X) = (x_i - X_{-i})/X^3$ is positive for $x_i < X_{-i}$ and it is negative for $x_i > X_{-i}$. Regardless of such a unique feature, Theorem 3 proves that if the number of rent-seeking agents is less than or equal to 4, the Nash equilibrium is always globally stable.
4. Conclusion

In this paper we have analyzed the global stability of Nash equilibrium and Nash-Cournot equilibrium in a rent-seeking game and Cournot oligopoly without product differentiation, respectively. Our simple stability analysis has become possible by characterizing the rent-seeking game and Cournot oligopoly without product differentiation as aggregative games in which each player’s payoff is a function of its choice of a strategy and the sum of choices of strategies of all players. It would be interesting to apply our general result in Theorem 1 to other aggregative games.
References


