On the equivalence of open-loop and feedback Nash equilibrium in an oligopoly on the market for a nonrenewable resource *

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Abstract

Starting from a full characterization of the open-loop Nash equilibrium in the cartel-versus-fringe model, we show that when the number of fringe firms is finite there is no closed-loop equilibrium that can result in the extraction path prescribed by the open-loop Nash equilibrium. However, in the limit case of an infinite number of purely price taking fringe firms we claim that the open-loop Nash and the feedback Nash equilibrium coincide.

Key words: nonrenewable resources, cartel-versus-fringe, Nash equilibrium, open-loop, feedback.
1 Introduction

The cartel-versus-fringe model of the oil market describes the pricing of oil in a situation where supply comes from a coherent cartel and a large group of fringe members. The model was introduced by Salant (1976), who considered the case of zero extraction costs and a continuum of price taking fringe members. He employed the open-loop Nash equilibrium (OLNE) as the equilibrium concept. The model was later analyzed by Ulph and Folie (1980), also with a continuum of fringe members and the OLNE equilibrium concept, but for positive constant marginal extraction costs, possibly differing between the cartel and the fringe. Lewis and Schmalensee (1980) and Loury (1986) have studied the case of a finite number of oligopolists. The former authors were mainly interested in the order of exploitation and their analysis mainly concerns the case of two players. Loury studies the case of equal costs. Recently Benchekroun et al. (2007) have provided a full characterization of the OLNE with a finite as well as an infinite number of fringe players, for all possible cost constellations, at least with constant marginal extraction costs.

In the literature there has been a debate regarding the subgame perfectness of OLNE. In the OLNE literature firms are assumed to choose an entire production path and therefore are able to commit to a production path at the initial time: they choose open-loop strategies. The alternative modelling choice is to allow the firms to adjust their extraction rates at each moment to the level of the stocks at that moment and to time: firms choose closed-loop strategies. Commitment can for example be achievable in the presence of a perfect futures market for the resource. In other circumstances information about the resource stocks of competitors may not be available. In that case modelling of each firm as taking an open-loop extraction strategy that maximizes this firm’s discounted profits is a reasonable assumption. The drawback is that if firms have information about all stocks at future dates and have the flexibility to adjust their production, the equilibrium obtained with open-loop strategies may not be subgame perfect.

Until now, this problem has been investigated only in discrete time, since it seems insurmountable to actually calculate the feedback equilibrium in continuous time. Polasky (1990) shows in a discrete time model with a finite number of players that the open-loop equilibrium is not subgame perfect if the exhaustion dates of firms
He then goes on to study a duopoly model with linear demand and equal and constant marginal extraction costs. He also postulates an exogenous instant of time $T$, after which the extracted commodity is worthless. He then claims that if the per period profit function is quadratic in extraction and depends only on current extraction (and not on existing stocks) and if no firm exhausts before $T$ open-loop and feedback equilibria coincide. But then he proves that in the duopoly model with equal initial stocks and equal constant marginal extraction costs the open-loop and the feedback equilibrium do not coincide because one firm can and will to manipulate its own exhaustion time in a profitable way. This possibility was overlooked by Eswaran and Lewis (1985) who claim that in the situation sketched, with identical players, linear demand and quadratic profit functions, the two equilibria coincide, based on the (incorrect) assumption that in such a situation the strategies of the players are equal.

The contribution of the present paper is twofold. First, we provide a general analysis of the equivalence of the equilibrium concepts in continuous time, based on previous work (Benchekroun et al. (2007)). We confirm Polasky’s claim that symmetry is not sufficient for the two equilibria to coincide. Since we work in continuous time our methodology differs from his. Our approach is to show first that a necessary condition for the equilibria to coincide is that the extraction rate of a firm only depends on its own stock. Then we represent the open-loop equilibrium extraction rates by means of functions of the existing stocks and determine whether the necessary condition is satisfied. The second contribution of the paper is to show that if it is assumed that the fringe members are pure price takers, open-loop and feedback will coincide.

Our methodology is closely related to the work done by Groot et al. (1992, 2003) who studied the case of the cartel being a Stackelberg leader and the fringe being a price taker. The cartel-versus-fringe model with Stackelberg leadership was first introduced by Gilbert (1978). It is well-know that in this model the open-loop Stackelberg equilibrium concept suffers from time inconsistency for plausible parameter values, and is therefore not a feedback equilibrium (see Newbery (1981) and Ulph (1982)).

In section 2 we present the model and the equilibrium concepts. In section 3 we fully characterize the open-loop Nash equilibrium. Section 4 treats the limit case
where the number of fringe firms is infinite thereby characterizing the equilibrium when the fringe firms are price takers. In section 5 we derive a necessary condition that a closed-loop Nash equilibrium must satisfy in order to generate the extraction path followed under the open-loop Nash equilibrium. We show that such a necessary condition is never satisfied. Section 6 treats the limit case where fringe firms are price takers.

2 Open-loop Nash equilibrium

There are two types of mines $c$ and $f$, distinguished by their marginal extraction costs. There is one $c$–type mine, owned by a cartel, and there are $n$ mines of the $f$–type. The owner of an $f$–mine is called a fringe member. Marginal extraction costs are constant: $k^c$ and $k^f$. The cartel’s initial stock is $S^c_0$. Fringe firm $i$ ($i = 1, 2, ..., n$) is endowed with an initial stock $S^f_{0i}$. Demand for the resource is stationary and linear with a choke price $\bar{p}: p(t) = \bar{p} - x(t)$, where $p(t)$ is the price at time $t$, $x(t)$ is demand at time $t$ and $\bar{p} > \max\{k^c, k^f\}$. We work in continuous time, which starts at time 0. Extraction rates at time $t \geq 0$ are denoted by $q^c(t) \geq 0$ and $q^f_i(t) \geq 0$. Define $q^f(t) = \sum_{i=1}^{n} q^f_i(t)$ and $S^f_0 = \sum_{i=1}^{n} S^f_{0i}$ as aggregate supply and initial stocks of the fringe firms. In an equilibrium at each moment $t \geq 0$ the price of the resource is given by $p(t) = \bar{p} - q^c(t) - q^f(t)$. For the time being all fringe firms are assumed identical with regard to their stocks: $S^f_{0i} = \frac{S^f_0}{n}$. Any extraction path for a firm is subject to the condition that total extraction over time equals the initial stock. This is called the resource constraint feasibility constraint. For the cartel it reads

$$\int_{0}^{\infty} q^c(s)ds = S^c_0$$

For fringe member $i$ it reads

$$\int_{0}^{\infty} q^f_i(s)ds = S^f_{0i}$$

We formulate the resource constraints as an equality because in any equilibrium all
resource stocks will get exhausted in view of the assumption that $\bar{p} > \max\{k^c, k^f\}$.

Firms are oligopolists in the resource market and the objective of each firm is to maximize the discounted sum of its profits with an equal and constant discount rate $r$.

**Definition**: Open-loop Nash Cournot equilibrium (OLNE)

A vector of functions $(q, p) \equiv (q^c, q^f_1, ..., q^f_n, p)$ with $(q(t), p(t)) \geq 0$ and $p(t) = \bar{p} - q^c(t) - q^f(t)$ for all $t \geq 0$ is an open-loop Nash-Cournot equilibrium if

i. the resource constraint is satisfied for all firms

ii.

$$\int_0^\infty e^{-rs}[\bar{p} - q^c(s) - q^f(s) - k^c]q^c(s)ds \geq \int_0^\infty e^{-rs}[\bar{p} - q^c(s) - q^f(s) - k^c]\dot{q}^c(s)ds$$

for all feasible $\dot{q}^c$.

iii. for all $i = 1, 2, ..., n$

$$\int_0^\infty e^{-rs}[\bar{p} - q^f_i(s) - q^c(s) - k^f]q^f_i(s)ds \geq \int_0^\infty e^{-rs}[\bar{p} - \sum_{j \neq i} q^f_j(s) - q^f_i(s) - q^c(s) - k^f]\dot{q}^f_i(s)ds$$

for all feasible $\dot{q}^f_i$.

Benchekroun et al. (2007) characterize the open-loop Nash equilibrium. They even consider a more general case, where there can also be an arbitrary number of cartel members. For our present purpose this is less relevant, as will be made clear in due course. By $S$, $C$ and $F$ we denote intervals of time with simultaneous supply, sole supply by the cartel and sole supply by the fringe, respectively. Benchekroun et al. have established the following propositions.

**Proposition 1**

i. Suppose

$$\frac{1}{2}(\bar{p} + k^c) < k^f$$

For a given $S^h_0$, there exists $\tilde{S}^l_0 > 0$ such that the equilibrium sequence reads $C \rightarrow S \rightarrow F$ if $S^c_0 > \tilde{S}^c_0$ and $S \rightarrow F$ if $S^c_0 \leq \tilde{S}^c_0$. 

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ii. Suppose
\[ \frac{1}{2} (\bar{p} + k^c) = k^f \]
Then the equilibrium reads \( S \to F \).

iii. Suppose
\[ \frac{1}{2} (\bar{p} + k^c) > k^f \]

iiiia. If
\[ \frac{S_0^c}{S_0^f / n} = \frac{\bar{p} + nk^f - (n + 1)k^c}{\bar{p} + k^c - 2k^f} \]
then the equilibrium reads \( S \).

iiiib. If
\[ \frac{S_0^c}{S_0^f / n} < \frac{\bar{p} + nk^f - (n + 1)k^c}{\bar{p} + k^c - 2k^f} \]
then the equilibrium reads \( S \to F \).

iiiic. If
\[ \frac{S_0^c}{S_0^f / n} > \frac{\bar{p} + nk^f - (n + 1)k^c}{\bar{p} + k^c - 2k^f} \]
then the equilibrium reads \( S \to C \).

\[ r \left( 2S_0^c + S_0^f \right) = (\bar{p} - k^c) \left( rT - 1 + e^{-rT} \right) \]

The case of an infinite number of fringe members and a single cartel can easily be treated as the limit for the number of fringe members going to infinity.

**Proposition 2.**

The open loop Nash equilibrium of the cartel-versus-fringe is characterized as follows:

i. If \( \frac{1}{2} (\bar{p} + k^c) < k^f \), then the Nash equilibrium sequence is \( C \to S \to F \), with the \( F \) phase collapsing if \( S_0^c \) is small.

ii. If \( \frac{1}{2} (\bar{p} + k^c) = k^f \), then the Nash equilibrium sequence is \( S \to F \).

iii. If \( \frac{1}{2} (\bar{p} + k^c) > k^f \), then the Nash equilibrium sequence is...
$S$ if $\frac{S^d_0}{S_0^0} = \frac{k_f - k^c}{\bar{p} + k^c - 2k_f}$

$S \rightarrow F$ if $\frac{S^d_0}{S_0^0} < \frac{k_f - k^c}{\bar{p} + k^c - 2k_f}$

$S \rightarrow C$ if $\frac{S^d_0}{S_0^0} > \frac{k_f - k^c}{\bar{p} + k^c - 2k_f}$

3 Open-loop versus closed-loop for a finite number of players

The definition of a closed-loop Nash equilibrium reads as follows. A closed-loop strategy for a firm is a decision rule that gives the extraction rate at $t$ as a function of $t$ and the vector of stocks at time $t$, $S (t) = (S^c (t), S^f_1 (t), S^f_2 (t), ..., S^f_n (t))$.

Definition: Closed-loop Nash-Cournot equilibrium (CLNE)

A vector of closed-loop strategies $\varphi \equiv (\phi^c, \phi^f_1, ..., \phi^f_n)$ is a closed-loop Nash-Cournot equilibrium if

i. the resource constraint is satisfied for all firms, where $q^c (t) = \phi^c (t, S (t))$ and $q^f_i (t) = \phi^f_i (t, S (t))$ ($i = 1, 2, ..., n$)

ii.

$$\int_0^\infty e^{-rs} [\bar{p} - \phi^c (t, S (t)) - \sum_{i=1}^n \phi^f_i (s, S (s)) - k^c] \phi^c (t, S (t)) ds$$

$$\geq \int_0^\infty e^{-rs} [\bar{p} - \sum_{i=1}^n \phi^f_i (s, S (s)) - \hat{\phi}^c (t, S (t)) - k^c] \hat{\phi}^c (t, S (t)) ds$$

for all feasible $\hat{\phi}^c$.

iii. for all $i = 1, 2, ..., n$
\[
\int_0^\infty e^{-rs}[\hat{p} - \sum_{j=1}^n \phi_j^f (s, S (s)) - \phi^c (t, S (t)) - k^f] \phi_i^f (s, S (s)) \, ds
\]

\[
\geq \int_0^\infty e^{-rs}[\hat{p} - \sum_{j \neq i}^n \phi_j^f (s, S (s)) - \phi_i^f (s, S (s)) - \phi^c (t, S (t)) - k^f] \phi_i^f (s, S (s)) \, ds
\]

for all feasible \( \phi_i^f \).

In the literature essentially two approaches to exploring the feedback equilibrium can be found. The first is to postulate a functional relationship between extraction rates and existing resource stocks, and then to see whether the Hamilton Jacobi Bellman (HJB) conditions are satisfied. This works in several cases (e.g., Salo and Tahvonen (2001), Eswaran and Lewis (1985)). However, no such simple relationship works in the particular model at hand with linear demand and constant but different marginal extraction rates. We will employ an alternative approach, described below.

**The case \( S \to F \)**

We know from Proposition 1 that if \( k^f > \frac{1}{2}[\hat{p} + k^c] \) the OLNE equilibrium gives the following sequence \( S \to F \), if the initial resource stock of the cartel is not too large. We seek to determine if there exists a feedback Nash equilibrium, that is therefore subgame-perfect, that replicates the exploitation path of the OLNE, given a vector of initial stocks. The cartel takes the closed-loop strategy of the fringe as given \( \phi^f (S, t) \) and chooses a closed-loop strategy \( \phi^c (S, t) \) that maximizes its discounted sum of profits

\[
\int_t^\infty e^{-r(s)} (\hat{p} - q^c(s) - \phi^f (S(s), s) - k^c) q^c(s) \, ds
\]

subject to

\[
\int_t^\infty q^c(s) \, ds \leq S^c
\]

and

\[
\int_t^\infty \phi_i^f (S(s), s) \, ds \leq S_i^f, \ i = 1, 2, ..., n
\]

for all non-negative couples \( (S, t) \), with \( q^c(s) = \phi^c (S(s), s) \).
The Hamiltonian associated with the cartel's problem is given by

\[ H^c(q^c, S, \mu^c, \mu^f, t) = e^{-rt} \left( \bar{p} - q^c - \sum_{i=1}^{n} \phi^f_i(S, t) - \mu^c_i(t) \right) q^c - \mu^c_i q^c - \sum_{i=1}^{n} \mu^f_i \phi^f_i(S, t) \]

where \( \mu^c_i \) is the costate variable associated with \( S^c \) and \( \mu^f_i \) is the costate variable associated with \( S^f_i \). Applying the maximum principle gives the following set of necessary conditions for an interior solution at time \( t \):

\[ e^{-rt} \left( \bar{p} - 2q^c(t) - \phi^f(S(t), t) - \mu^c(t) \right) - \mu^c(t) = 0 \]

\[ \mu^c_i(t) = -\frac{\partial H^c}{\partial S^c_i} = \sum_{i=1}^{n} \left( e^{-rt} q^c(t) + \mu^c_i(t) \right) \frac{\partial \phi^f_i(S(t), t)}{\partial S^c_i} \]

\[ \mu^f_i(t) = -\frac{\partial H^c}{\partial S^f_i} = \sum_{i=1}^{n} \left( e^{-rt} q^c(t) + \mu^c_i(t) \right) \frac{\partial \phi^f_i(S(t), t)}{\partial S^f_i} \]

where

\[ \phi^f(S(t), t) = \sum_{i=1}^{n} \phi^f_i(S(t), t) \]

Appendix A provides a further characterization of the OLNE in this case, based on Benchekroun et al. (2007). There it is shown that along the phase of simultaneous supply from time \( t_0 \) till time \( t_1 \) the production paths of the fringe and the cartel along the OLNE are given by

\[(n + 2)q^c(t) = \bar{p} + n \left( k^f + \lambda^f e^{rt} \right) - (n + 1) \left( k^c + \lambda^c e^{rt} \right) \quad (4)\]

\[ \frac{2 + n}{n} q^f(t) = \bar{p} + \left( k^c + \lambda^c e^{rt} \right) - 2 \left( k^f + \lambda^f e^{rt} \right) \quad (5)\]

where \( \lambda^c \) and \( \lambda^f \) are the constant shadow prices of the resource stocks of the cartel and the fringe members respectively. Hence, for a CLNE to result in the extraction path of the OLNE, we must have \( \mu^c_c(s) = \lambda^c \), for all instants \( s \geq t \) for all \( t \geq 0 \). From the necessary conditions it follows that then

\[ \sum_{i=1}^{n} \left( e^{-rs} q^c(t) + \mu^c_i(t) \right) \frac{\partial \phi^f_i(S(t), t)}{\partial S^c} = 0 \]

Given the symmetry of fringe firms we must have either \( e^{-rs} q^c + \mu^f_i = 0 \) where \( q^c \) is the OLNE equilibrium path of the cartel and therefore \( \mu^f_i(t) = -e^{-rt} q^c(t) \) or
\[ \frac{\partial \phi^f(S(t), t)}{\partial S^c} = 0. \] The first condition is in contradiction with the necessary conditions since it implies that \( \mu^c_{f_1} = 0 \), but \( e^{-rs}q^c(s) \) is not constant along the OLNE. Given the symmetry of fringe firms we use

\[
\phi^f_i(S(t), t) = \frac{\phi^f_i(S(t), t)}{n} = \frac{q^f(t)}{n}
\]

So

\[
2 + n \frac{\partial \phi^f_i(S(t), t)}{\partial S^c} = \frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} e^{ rt }\]

where

\[ \lambda^f = e^{- rt} (\bar{p} - k^f) \]

and

\[ \lambda^c = \frac{n}{n + 1} \lambda^f + \frac{e^{- r t_1} (\bar{p} + nk^f - (n + 1) k^c)}{n + 1} \]

As explained in appendix A the first of these two latter equations states that the market price at the instant of exhaustion of the resource equals the choke price; the second equation states that the price path is continuous. The two equations give

\[ \lambda^c - 2\lambda^f = \left( \frac{n}{n + 1} - 2 \right) e^{- rt} (\bar{p} - k^f) + \frac{1}{n + 1} e^{- r t_1} (\bar{p} + nk^f - (n + 1) k^c) \quad (6) \]

The time of transition \( t_1 \) and the final time \( T \) satisfy (see appendix A):

\[ (2 + n) r S_0^c = (\bar{p} + nk^f - (n + 1) k^c) (rt_1 - 1 + e^{- r t_1}) \quad (7) \]

\[ r \left( S_0^f + \frac{n}{n + 1} S_0^c \right) = \frac{n}{n + 1} (\bar{p} - k^f) (rT - 1 + e^{- r T}) \quad (8) \]

From (6) we have

\[ \frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} = -r \left( \frac{n}{n + 1} - 2 \right) e^{- rt} \frac{\partial T}{\partial S^c} (\bar{p} - k^f) - re^{- r t_1} \frac{\partial t_1}{\partial S^c} \left( \frac{\bar{p} + nk^f - (n + 1) k^c}{n + 1} \right) \quad (9) \]

We get \( \frac{\partial T}{\partial S^c} \) and \( \frac{\partial t_1}{\partial S^c} \) from (7) and (8):

\[
(2 + n) r = (\bar{p} + nk^f - (n + 1) k^c) (rt_1 - re^{ rt - r t_1 }) \frac{\partial t_1}{\partial S^c}
\]

and
\[
0 = -\frac{n}{n+1} + \frac{n}{n+1} (\bar{p} - k^f) (rT - re^{rt-rT}) \frac{\partial T}{\partial S^c}
\]

Hence
\[
(\bar{p} - k^f) \frac{\partial T}{\partial S^c} = \frac{1}{(T - e^{rt-rT})}
\]

and
\[
(\bar{p} + nk^f - (n+1)k^c) \frac{\partial t_1}{\partial S^c} = \frac{(2+n)}{t_1 - e^{rt-rT}}.
\]

Substituting gives
\[
\frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} = -r \left( \frac{n}{n+1} - 2 \right) e^{-rT} \frac{\partial T}{\partial S^c} (\bar{p} - k^f) - re^{-rt_1} \frac{\partial t_1}{\partial S^c} \frac{(\bar{p} + nk^f - (n+1)k^c)}{n+1}
\]
or
\[
\frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} = -r \left( \frac{n}{n+1} - 2 \right) e^{-rT} \frac{1}{(T - e^{rt-rT})} - \frac{1}{n+1} \frac{(2+n)}{t_1 - e^{rt-rT}}
\]

or
\[
\frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} = r \left( \frac{n+2}{n+1} \right) e^{-rT} \frac{1}{(T - e^{rt-rT})} - \frac{e^{-rt_1}}{(t_1 - e^{rt-rT})}
\]

or
\[
\frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} = r \left( \frac{n+2}{n+1} \right) \left( \frac{1}{(Te^{rt}-e^{rt})} - \frac{1}{(t_1 e^{rt_1} - e^{rt})} \right)
\]

For any \( t \) we have
\[
f(X) = \frac{1}{(Xe^{rX} - e^{rt})}
\]

strictly decreasing in \( X \) and therefore \( \frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} \neq 0 \) since \( T > t_1 \). This implies that the necessary condition for the CLNE to yield the OLNE extraction path is not met.

We have thus shown that for any equilibrium that reads \( C \rightarrow S \rightarrow F \) or \( S \rightarrow F \) the necessary condition for the CLNE to yield the OLNE extraction path is not met. The argument goes trough for any cost constellation that yields this equilibrium sequence.

Note that our result is also true even in the limit case where \( n = \infty \) since for \( n \rightarrow \infty \) we have
\[
\frac{\partial (\lambda^c - 2\lambda^f)}{\partial S^c} = r \left( \frac{1}{(Te^{rt}-e^{rt})} - \frac{1}{(t_1 e^{rt_1} - e^{rt})} \right) \neq 0
\]

and therefore the limit case where \( n = \infty \) the CLNE does not coincide with the cartel-fringe outcome.
The case $S \to C$

We know from Proposition 1 that if $k^c > \frac{1}{2} [\bar{p} + k^f]$ the equilibrium reads $S \to C$ if the initial resource stock of the fringe is not too large. We seek to determine whether there exists a feedback Nash equilibrium, that is therefore subgame-perfect, that replicates the exploitation path of the OLNE, given a vector of initial stocks. Along the phase of simultaneous supply we have

$$(n + 2) q^c(t) = \bar{p} + n \left( k^c + \lambda^c e^{rt} \right) - (n + 1) \left( k^f + \lambda^f e^{rt} \right)$$

$$\frac{2 + n}{n} q^f(t) = \bar{p} + \left( k^f + \lambda^f e^{rt} \right) - 2 \left( k^c + \lambda^c e^{rt} \right)$$

Moreover

$$\lambda^c = e^{-rt} (\bar{p} - k^c)$$

$$\frac{1}{2} (\bar{p} + k^c + e^{rt_1} \lambda^c) = k^f + e^{rt_1} \lambda^f$$

and

$$\frac{2 + n}{n} r S^f_0 = (\bar{p} + k^c - 2k^f) \left( rt_1 - 1 + e^{-rt_1} \right)$$

$$r \left( S^f_0 + \frac{1}{2} S^f_0 \right) = (\bar{p} - k^c) \left( rT - 1 + e^{-rT} \right)$$

Therefore, $q^f(t)$ is independent of $S^c$. Contrary to the previous case we will henceforth concentrate on the fringe. The problem is that we cannot just repeat the steps taken in the previous case, since we have to be clear about what to mean by a marginal change in the stock of one of the fringe members, keeping the other stocks fixed. This poses a difficulty because it has been assumed that all fringe members are equal, and the OLNE has been derived under that assumption. However, it is not difficult to conceptualize what will happen if one fringe member is given an additional reserve. All other fringe members will exhaust their resource before this fringe member under consideration does, as is demonstrated in Appendix B. Therefore it is left with the cartel as sole competitor. We are therefore done if we can show that the OLNE and the CLNE do not coincide for the case of a single cartel and a single fringe member.
Due to symmetry this is straightforward since we can just repeat the steps taken in the previous case, ceteris paribus, and obtain the same negative result. For the sake of completeness the proof is given in detail in appendix C.

4 Open-loop versus closed-loop for an infinite number of fringe members

It was shown in the previous section that OLNE and CLNE do not coincide for a finite number of players, and even not in the limit for the number of fringe members going to infinity. The main reason was that we assumed that fringe members, even if they are small, can condition their actions on the existing stocks. Now, one could argue that the OLNE for an infinite number of fringe members can be seen as game between a single cartel, taking the supply by the fringe as given, and a group of price taking fringe members. For the open loop it makes no difference whether we assume price taking or take the limit for the number of fringe members going to infinity. However, for the analysis of the closed loop it does make a difference, as will be made clear below. In fact we argue that with a price taking fringe, open loop and closed loop coincide.

Price taking in a closed loop setting is a modelling choice and the literature is silent on this issue. It can be motivated by the assumed degree of sophistication of the firms and their farsightness as well as the possibility to commit to a production plan. Here we assume that the strategy of a fringe firm consists of a production path whereas the strategy of the dominant firm may be an open loop production path or closed-loop production rule. This modelling choice can be justified by assuming that firms condition their actions on payoff relevant information. In the case where a firm is price taker, the only relevant information is its stock. But this information is solely determined by its own past extraction. Choosing an extraction path or an extraction rule that depends on one own’s resource stock yields the same outcome. However, for a dominant firm, it is no longer true. Since a dominant firm determines the impact of its production decision on the price path, it is aware that at any given moment the future price path is determined by the extraction rates of all firms which conditioned on the available stocks to all firms. Therefore the information about available stocks
is a priori useful to the dominant firm when setting the extraction rate at a given moment.

The Hamilton-Bellman-Jacoby equation for the cartel reads

\[
\frac{\partial V_c}{\partial t} + \max_{q_c} \left[ e^{-rt}(\bar{p} - q^c - q^f - k^c)q^c - \frac{\partial V_c}{\partial S_f} q^f - \frac{\partial V_c}{\partial q^c} q^c \right] = 0
\]

Along an \( S \)-phase the maximization with respect to \( q^c \) yields

\[
q^c(t) = \frac{1}{2}(\bar{p} - k^c - e^{rt} \frac{\partial V_c}{\partial q^c} - q^f(t))
\]

For \( n \to \infty \) we have for the open-loop path

\[
q^f(t) = \bar{p} + k^c + \lambda^c e^{rt} - 2(k^f + \lambda^f e^{rt})
\]

Moreover \( \frac{\partial V^l}{\partial S^l} = \lambda^c \). Hence \( p(t) = k^f + \lambda^f e^{rt} \). It is also easily checked that the Hamilton-Bellman-Jacoby condition is satisfied. Therefore, with a price taking fringe open-loop and closed-loop coincide

\section{Conclusions}

In the cartel-versus-fringe model with linear demand and constant marginal extraction costs there exists no closed-loop Nash equilibrium that replicates the extraction path of the open-loop equilibrium, when the number of firms is finite. This confirms an earlier finding by Polasky (1990) for the case of a duopoly and discrete time. This result also holds when we take the limit of fringe members going to infinity. In the limit case of an infinite number of fringe firms we therefore claim that the open-loop Nash and the feedback Nash equilibrium don’t coincide either. However, if we assume price taking on the part of the fringe from the outset, then the two equilibrium concepts give rise to the same outcome. The limit of the open-loop equilibrium for the number of fringe members going to infinity yields price taking. However, the feedback equilibrium with a finite number of players does not converge to the feedback equilibrium under price taking.
6 References


Appendix A

Here we summarize the findings on the open-loop Nash equilibrium with a finite number of players. Each fringe firm $i$ takes the strategy profile of its $n$ competitors as given and maximizes its present value profits subject to the resource constraint. The corresponding Hamiltonian reads

$$H_i^f(q_i^f, \lambda_i^f, q^f, t) = e^{-rt} \left( \bar{p} - q^c - q^f - k^f \right) q_i^f + \lambda_i^f (-q_i^f)$$

where $q^f$ and $q^c$ denote the aggregate supply by the fringe and the supply by the cartel respectively. For the cartel the Hamiltonian reads

$$H_c(q^c, \lambda^c, q^f, t) = e^{-rt} \left( \bar{p} - q^c - q^f - k^f \right) q^c + \lambda^c (-q^c)$$

Among the necessary conditions we have that the co-state variables are constant since stocks are absent from the Hamiltonians. In addition the Hamiltonians are maximized with respect to the own supply of the agent. We will use the symmetry among the fringe players, i.e. $q_i^f = q^f/n$ and $\lambda_i^f = \lambda^f$ for all $i$. Then we arrive at the following necessary conditions.

Along an $F$ interval:

$$e^{-rt} \left( \bar{p} - q^f(t) - \frac{1}{n} q^f(t) - k^f \right) = \lambda^f$$

$$p(t) = \frac{1}{n + 1} \left( \bar{p} + n \left( k^f + \lambda^f e^{rt} \right) \right) \leq k^c + e^{rt} \lambda^c$$

The first condition follows from the maximization of the Hamiltonian of player $i$. The second condition is necessary in order for the cartel not to supply.

Along a $C$ interval:

$$e^{-rt} \left( \bar{p} - 2q^c(t) - k^c \right) = \lambda^c$$

$$p(t) = \frac{1}{2} \left( \bar{p} + k^c + \lambda^c e^{rt} \right) \leq k^f + e^{rt} \lambda^f$$

Along an $S$ interval
\[(2 + n)q^c(t) = \bar{p} + n \left( k^f + \lambda e^{rt} \right) - (n + 1) \left( k^c + \lambda e^{rt} \right) \]
\[\frac{n + 2}{n} q^f(t) = \bar{p} + k^c + \lambda e^{rt} - 2 \left( k^f + \lambda e^{rt} \right) \]
\[p(t) = \frac{1}{2 + n} \left( \bar{p} + k^c + \lambda e^{rt} + n(k^f + \lambda e^{rt}) \right) \]

Continuity of the price path at the different possible transitions gives:
- a transition at \( t \) from \( S \) to \( C \) or vice versa requires
  \[\frac{1}{2} (\bar{p} + k^f + \lambda e^{rt}) = k^f + \lambda e^{rt}\]
- a transition at \( t \) from \( S \) to \( F \) or vice versa requires
  \[\frac{1}{n + 1} (\bar{p} + n(k^f + \lambda e^{rt})) = k^c + \lambda e^{rt}\]
- a transition at \( t \) from \( F \) to \( C \) or vice versa requires
  \[\frac{1}{2} (\bar{p} + k^c + \lambda e^{rt}) = \frac{1}{n + 1} (\bar{p} + n(k^f + \lambda e^{rt}))\]

We also have to take into account that at the moment of exhaustion of all resource stocks, the price must have reached the choke level:

\[p(T) = \bar{p}\]

Finally, consider the sequence \( S \rightarrow C \), with \( C \) the final phase before exhaustion and where the transition takes place at instant of time \( t_1 \) and exhaustion at \( T \). Then it is tedious but straightforward to derive (see Benchekroun et al. (2007))

\[\frac{2 + n}{n} r S^f_0 = (\bar{p} + k^c - 2k^f) \left( r t_1 - 1 + e^{-r t_1} \right)\]
\[(2 + n) r S^c_0 = -\frac{1}{2} n \left( \bar{p} + k^c - 2k^f \right) \left( r t_1 - 1 + e^{-r t_1} \right) + (1 + \frac{1}{2} n) (\bar{p} - k^c) \left( r T - 1 + e^{-r T} \right)\]
For the sequence $S \to F$ we have

$$(2 + n)rS_0^c = (\bar{p} + nk^f - (n + 1)k^c)(rt_1 - 1 + e^{-rt_1})$$

$$\frac{2 + n}{n}rS_0^f = -\frac{1}{n + 1}(\bar{p} + nk^f - (n^f + 1)k^c)(rt_1 - 1 + e^{-rt_1}) + \frac{2 + n}{n + 1}(\bar{p} - k^f)(rT - 1 + e^{-rT})$$

### Appendix B

In this appendix we modify the problem discussed in appendix A so as to allow for a fringe member with a larger stock than all other $n$ fringe members. We will show that the stocks of all other fringe members will be depleted before the stock of this particular fringe member is. The variables referring to the larger fringe member are denoted by upper bars. Among the necessary conditions for an OLNE we have

$$e^{-rt}(\bar{p} - 2\bar{q}^f(t) - \bar{q}^c(t) - q^f(t) - k^f) \leq \bar{\lambda}^f$$

$$e^{-rt}(\bar{p} - \bar{q}^f(t) - q^c - \frac{n + 1}{n}q^f(t) - k^f) \leq \lambda^f$$

$$e^{-rt}(\bar{p} - \bar{q}^f(t) - 2q^c - q^f(t) - k^c) \leq \lambda^c$$

with equality holding if $\bar{q}^f(t), q^f(t)$ and $q^c(t)$ are positive, respectively. Since the fringe members only differ with respect to the stocks, it is clearly the case that the shadow price of the larger stock is smaller that the shadow price of each smaller stocks: $\bar{\lambda}^f < \lambda^f$. This fact immediately implies that we cannot have simultaneous supply at the end because that would imply

$$\bar{p} = k^f + e^{rT}\bar{\lambda}^f = k^f + e^{rT}\lambda^f = k^c + e^{rT}\lambda^c$$

which violates the requirement. Moreover, it cannot be the case that the larger stock is exhausted before the smaller stock, because that would require that

$$e^{-rT}(\bar{p} - q^c(T) - q^f(T) - k^f) \leq \bar{\lambda}^f$$

$$e^{-rT}(\bar{p} - q^c - \frac{n + 1}{n}q^f(T) - k^f) \leq \lambda^f$$
at the time $T$ of exhaustion of the larger stock, which is infeasible.

**Appendix C**

Here we prove that the case $S \rightarrow C$ cannot be sustained as a closed-loop equilibrium. As was made clear in the main text, we only have to consider the case of a single fringe member.

The cartel takes the closed-loop strategy of the fringe as given $\phi^f(S, t)$ and chooses a closed-loop strategy $\phi^c(S, t)$ that maximizes its discounted sum of profits

$$\int_{t}^{\infty} e^{-rs} \left( \bar{p} - q^c(s) - \phi^f(S(s), s) - k^c \right) q^c(s) ds$$

subject to

$$\int_{t}^{\infty} q^c(s) ds \leq S^c$$

and

$$\int_{t}^{\infty} \phi^f(S(s), s) ds \leq S^f$$

for all non-negative couples $(S, t)$, with $q^c(s) = \phi^c(S(s), s)$. The Hamiltonian for the cartel reads

$$H^c(q^c, S, \mu^c, \mu^f) = e^{-rt} \left( \bar{p} - q^c - \phi^f(S(t), t) - k^c \right) q^c - \mu^c q^c - \mu^f \phi^f(S, t)$$

where $\mu^c$ is the costate variable associated with $S^c$ and $\mu^f$ is the costate variable associated with $S^f$. Applying the Maximum Principle gives the following set of necessary conditions for an interior solution (i.e. $q^f > 0$ and $q^c > 0$):

$$e^{-rt} \left( \bar{p} - 2q^c(t) - \phi^f(S(t), t) - k^c \right) - \mu^c(t) = 0$$

$$\dot{\mu}^c(t) = -\frac{\partial H^c}{\partial S^c} = (e^{-rt} q^c(t) + \mu^f(t)) \frac{\partial \phi^f(S(t), t)}{\partial S^c}$$

$$\dot{\mu}^f(t) = -\frac{\partial H^c}{\partial S^f} = (e^{-rt} q^c(t) + \mu^c(t)) \frac{\partial \phi^f(S(t), t)}{\partial S^f}$$

We consider the case where the OLNE consists of a final phase with $S \rightarrow C$. The Hamiltonian associated with the OLNE problem of firm $j$ ($j = c, f$) reads
\[ H^j(q^j, \lambda^j, t) = e^{-rt} \left( \bar{p} - q^c - q^f - k^j \right) q^j + \lambda^j (-q^j) \]

Among the necessary conditions we have that the co-state variable \( \lambda^j \) is constant. In addition the Hamiltonian is maximized. This implies that if at time \( t \) there is simultaneous supply we have

\[ 3q^c(t) = \bar{p} + k^f + \lambda^c e^{rt} - 2 \left( k^c + \lambda^c e^{rt} \right) \]

\[ 3q^f(t) = \bar{p} + \left( k^c + \lambda^c e^{rt} \right) - 2 \left( k^f + \lambda^f e^{rt} \right) \]

\[ 3p(t) = \bar{p} + k^c + \lambda^c e^{rt} + k^f + \lambda^f e^{rt} \]

Along the \( C \) interval we have

\[ 2q^c(t) = \bar{p} - k^c - \lambda^c e^{rt} \]

\[ 2p(t) = \bar{p} + k^c + \lambda^c e^{rt} \]

In addition, the equilibrium price is continuous at the time of transition \( t_1 \). Moreover, at the final time \( T \) the price must be equal to \( \bar{p} \). Taking this into account we can derive the stocks needed to have this equilibrium from some \( t \) in the \( S \)-phase on. We end up with the following set of equations at such an instant of time.

\[ 3q^c(t) = \bar{p} + \left( k^f + \lambda^c e^{rt} \right) - 2 \left( k^c + \lambda^c e^{rt} \right) \]

\[ \lambda^c = e^{-rT} \left( \bar{p} - k^f \right) \]

\[ \frac{1}{3} (\bar{p} + k^c + \lambda^c e^{rt_1} + k^f + \lambda^c e^{rt_1}) = \frac{1}{2} (\bar{p} + k^f + \lambda^f e^{rt_1}) \]

\[ (C1) \quad 3rS^f(t) = \left( \bar{p} + k^c - 2k^f \right) (rt_1 - rt - 1 + e^{rt-rt_1}) \]

\[ (C2) \quad 3rS^c(t) = -\frac{1}{2} \left( \bar{p} + k^c - 2k^f \right) (rt_1 - rt - 1 + e^{rt-rt_1}) \]

\[ + \frac{3}{2} (\bar{p} - k^c) (rT - rt - 1 + e^{rt-rT}) \]
For a CLNE to result in the extraction path of the OLNE, we must therefore have
\( \mu_c^c(s) = \lambda^c \) for all instants \( s \geq t \) for all \( t \geq 0 \). Therefore \( \mu_c^c \) is constant. It follows that then
\[
(\exp(-rt)q^c(t) + \mu^c_j) \frac{\partial \phi^c}{\partial S^c} (S(t),t) = 0
\]
This implies that either (i) \( \exp(-rt)q^c(t) + \mu^c_j(t) = 0 \) where \( q^c \) is the OLNE equilibrium path of the cartel and therefore \( \mu^c_j(t) = -\exp(-rt)q^c(t) \) or (ii)
\[
\frac{\partial \phi^c}{\partial S^c} (S(t),t) = 0.
\]

Condition (i) implies that \( \mu^c_j = 0 \), but \( \exp(-rt)q^c(t) \) is not constant along the OLNE. Therefore, for a CLNE to result in the extraction path of the OLNE, we must have
\[
\frac{\partial (\phi^c (S(t),t))}{\partial S^c} = 0
\]
along the OLNE paths of the stock where there is simultaneous supply. We next show that this condition is not met in the open-loop Nash equilibrium.

Our strategy is to assume that the open-loop equilibrium is subgame perfect and consequently represent extraction by the cartel as a function of time and the existing stocks. So, we first write
\[
3 \frac{\partial \phi^c}{\partial S^f} = \frac{\partial (\lambda^f - 2\lambda^c)}{\partial S^f} \exp(rt)
\]
We have
\[
\lambda^f - 2\lambda^c = -\frac{3}{2} \exp(-rT) (\bar{p} - k^c) + \frac{1}{2} \exp(-rt_1) (\bar{p} + k^c - 2k^f)
\]
So
\[
\frac{\partial (\lambda^f - 2\lambda^c)}{\partial S^f} = r \frac{3 \exp(-rT)}{2} \frac{\partial T}{\partial S^f} (\bar{p} - k^c) - r \frac{\partial t_1}{\partial S^f} \frac{\exp(-rt_1)}{2} (\bar{p} + k^c - 2k^f)
\]
The derivatives with respect to the stocks from (C1) and (C2)
\[
3r = (\bar{p} + k^c - 2k^f) (rt_1 - \exp(rt_1-t_1)) \frac{\partial t_1}{\partial S^f}
\]
\[
0 = -\frac{1}{2} (\bar{p} + k^c - 2k^f) (rt_1 - \exp(rt_1-t_1)) \frac{\partial t_1}{\partial S^f} + \frac{3}{2} (\bar{p} - k^c) (rT - \exp(rT-t)T) \frac{\partial T}{\partial S^f}
\]
Therefore
\[ 0 = -\frac{3}{2} r + \frac{3}{2} (\bar{p} - k^c) (rT - e^{rt-rT}) \frac{\partial T}{\partial S_f} \]

Or

\[ \frac{1}{(T - e^{rt-rT})} = (\bar{p} - k^c) \frac{\partial T}{\partial S_f} \]

and

\[ \frac{3}{(t_1 - e^{rt-rt_1})} = (\bar{p} + k^c - 2k^f) \frac{\partial t_1}{\partial S_f} \]

Substituting gives

\[
\frac{\partial (\lambda^f - 2\lambda^c)}{\partial S_f} = r \frac{3e^{-rT}}{2} \frac{1}{(T - e^{rt-rT})} - r \frac{3}{(t_1 - e^{rt-rt_1})} \frac{e^{-rt_1}}{2} \\
= \frac{3r}{2} \left( \frac{e^{-rT}}{(T - e^{rt-rT})} - \frac{e^{-rt_1}}{(t_1 - e^{rt-rt_1})} \right) \\
= \frac{3r}{2} \left( \frac{1}{(Te^{rt} - e^{rt})} - \frac{1}{(t_1e^{rt_1} - e^{rt})} \right)
\]

For any \( t \) we have

\[ f(X) = \frac{1}{(Xe^{rX} - e^{rt})} \]

strictly decreasing in \( X \) and therefore

\[ \frac{\partial (\lambda^f - 2\lambda^c)}{\partial S^c} \neq 0 \]